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ON THE PROMISES AND PITFALLS OF COLLABORATIVE LEARNING WITH MULTIPLE REPRESENTATIONS

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Abstract

Multiple representations of function have provided the foundation for a variety of mathematics curricula and novel technology-based learning environments that illustrate the rich conceptual linkages among symbols, graphs, tables and situations. This paper presents a designed learning environment intended to build on the affordances of representations as tools for both reasoning and communication. The design assigns unique representations to each member of a group in order to engage students in learning about the relationships among multiple representations as they work together on a shared task. By analyzing two teams of students working together in this environment over several weeks, we sought to understand the conditions for success in designs for collaboration with multiple representations. To that end, we pursued two research questions: 1) How do the developing problem-solving strategies of groups draw on various elements in a distributed array of function and problem representations? 2) How do learners work together to establish ways of interpreting multiple and distributed representational tools as they develop and enact collaborative strategies? Over the course of several extended problem-solving sessions, each group developed several successive alignments of participants and representations as they learned to solve increasingly difficult tasks. Our findings highlight the emergent and often unexpected meanings that learners established for representational tools as their groups reorganized into increasingly effective problem-solving ensembles. We note some promise and also several challenges in the orchestration of collaborative student interactions with multiple representations. Regarding the promise, our findings echo those of prior research regarding learners' considerable competence and creativity in interpreting and applying distributed representational tools, as well as the careful coordination among learners involved in establishing and acting on those interpretations. Challenges in this design space include instances in our data

where students capitalized on connections among representations without really trying to understand those connections, temporarily undermined the distributed character of the representations by working together on one representation, and worked more efficiently by reducing the number of participants actively involved in breaking codes. Our findings indicate that managing these challenges requires presenting student groups with regular opportunities to reconsider and reorganize their roles, and to experiment with different meanings and uses of flexible tools in the context of tasks with carefully sequenced levels of difficulty.

INTRODUCTION

Computers and graphing calculators have been widely heralded for their potential to support student learning by linking multiple representations of mathematical problems, relationships and phenomena (Kaput, 1992). Keeping pace with these innovations, new mathematics software, curricular materials and instructional approaches emphasizing multiple representations have proliferated in recent decades (see, for example, Burrill, 1992; Fey, 1989; Kaput, 1989; Moschkovich, Schoenfeld & Arcavi, 1993; Pea, 1987; Roschelle, Pea, Hoadley, Gordin, & Means, 2000; Yerushalmy & Shternberg, 2001). Likewise, recent practitioner literature in mathematics education is rich with examples of ways multiple representations might serve as resources for supporting students' conceptual understanding (e.g., Clement, 2004; Tripathi, 2008), transforming teaching practices (Rider, 2007), promoting equitable instruction and incorporation of multiple learning styles (Lesser, 2000), developing students' use of mathematical language (Herbel-Eisenmann, 2002), and structuring cooperative learning activities (Cleaves, 2008).

But the very richness and variety of these environments and activities and the representations they comprise make the nature and the distinct promise of learning in such settings difficult to specify. As Kaput (1998) observed, the subject of representations in mathematics education "is notorious for its complexity and its subtlety because it seems to connect to everything we want to know or study" (p. 266). External representations such as graphs, tables and symbolic expressions are fundamental tools of mathematical activity, providing both conceptual resources for organizing and analyzing information in problem-solving processes, and social resources for communicating ideas and coordinating interaction. And representations never stand alone; the utility of a representation rests precisely in its capacity to establish and to capitalize on relations and correspondences among multiple elements in a system of meaning (Goldin & Shteingold, 2001). Moreover, the emergence of computational media has made possible not only a wide array of novel representational forms, but also new ways of communicative sharing and establishing of connections among representations (Roschelle, 1996). So the study of representations is inevitably the study of semiotic linkages: between representing sign and represented referent, between different representations of a common phenomenon, between different representational media, between participants in representation-mediated activity.

This paper investigates these representational linkages as they emerge in the context of a novel learning environment. In particular, we explore the reciprocal relations between multiple representations and multiple learners in a design for collaborative classroom problem solving. As an instance of design research (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003), this paper seeks to engage intersections between learning theory and learning environment design. We introduce a learning environment with properties designed to serve collaborative learning with multiple linked representations. We consider the ways that multiple linked representations might serve as resources for organizing collaboration, and how collaboration with multiple linked representations might occasion new insights into forms of reasoning and interaction mediated by representational tools. We begin by considering key perspectives on representation and multiple representations in learning and then present a design for a learning environment that seeks to integrate insights from those different perspectives, in particular by linking multiple representations of a mathematical function across the devices of multiple students through a classroom network of wireless handheld computers. We then report on a study that investigated the ways in which this design for multiple linked devices and representations supported students' strategic engagement with and collaborative participation in mathematics problem solving tasks.

We document the ways learners appropriated that design in practice, and then consider the implications for uses of socio-technical systems such as these for fostering mathematical learning.

Reasoning and Interacting with Multiple Representational Tools

The practices of mathematical reasoning are replete with representational artifacts such as symbols, equations, graphs, tables, and diagrams. These artifacts are representations in the sense that they stand in for something else: a set of abstract objects or real-world phenomena, aspects of those phenomena, or relationships among objects. Representations, then, pair a representational artifact with a represented referent through some convention or interpretation that establishes the mapping between the two. Peirce, in his pioneering theory of representations (or "signs": see Peirce et al., *Collected Papers*, 1935), terms this determination of relations between sign/representation and object/referent an "interpretant," the third part in the triadic model of a semiotic system. Becoming familiar with conventions for interpreting, applying and reproducing standard representational forms such as algebraic expressions and Cartesian graphs, as well as practicing the construction of new or nonstandard ways to represent ideas and information, are each fundamental aspects of learning mathematics (Greeno & Hall, 1997).

A given mathematical idea can be represented in a variety of modes; different representations of the same phenomenon can highlight different features or characteristics, and so provide distinct conceptual resources and problem-solving affordances in the context of a particular task (Ainsworth, 1999; Larkin & Simon, 1987; Parnafes & DiSessa, 2004). Learning to represent the same concept in different ways, and to move fluently between multiple representational modes, is thus an important element of representational fluency; the National Council of Teachers of Mathematics (NCTM) Representation standard, for example, stresses the importance of opportunities for students to “select, apply, and translate among mathematical representations” (NCTM, 2000). Representational fluency in reasoning is a more general hallmark of expertise in many content domains beyond mathematics, such as the sciences (National Research Council, 2000). A wide variety of computer- and handheld-calculator-based instructional tools developed over the last two decades have been aimed at providing those opportunities by dynamically linking several representations so that changes made to one propagate corresponding changes in the others. These linked multiple representations and technologies for their simultaneous display have been particularly important in the study of functions, and in a number of instructional approaches to algebra and to the secondary curriculum in which functions play a central and organizing role (Kaput, 1992; see Kieran & Yerushalmy, 2004). The same functional relationship can be readily expressed through an algebraic equation, a Cartesian graph, a table of numerical values, a set of ordered pairs, or a written or verbal description (Kaput, 1989; Moschkovich et al., 1993; Yerushalmy, 2006).

Despite the apparent promise of engaging students with multiple representations, early studies indicated the troubles students may have in making connections between different representational modes, particularly the symbolic and the graphical, even after considerable experience with computer-based learning environments which dynamically link these function representations (Schoenfeld, Smith, & Arcavi, 1993; Yerushalmy, 1991). Learners' difficulties reveal fundamental challenges in making sense of multiple representations of a function. Although educators may believe that they simply ‘perceive’ the same function to be at once represented by a table and a graph, a student viewing those same linked representations has to *learn* those equivalences, and may thus not recognize the continuity of that mathematical function across its different representational modes (Thompson & Sfard, 1994).

On the other hand, learners' difficulties in translating among different representations may reflect the challenges associated with making sense of any of the single function representations. For example, research on students' learning about symbolic expressions (e.g. Kieran, 1992) and graphs (e.g. Leinhardt, Zaslavsky & Stein, 1990) has documented the considerable complexity posed to introductory algebra students by each of these representational modes. Evidence of learners' difficulties in making standard interpretations of canonical representational forms or the links among multiple representational modes may thus reveal far less about the effectiveness of a learning environment or any learner deficiencies than it does about the significant interpretive work required to establish the meaning of any mathematical representation (Lee & Sherin, 2006). A growing body of research indicates that learners bring considerable capabilities for building and reasoning with representations into instructional settings (e.g. diSessa, Hammer, Sherin & Kolpakowski, 1991; Enyedy, 2005; Izsak, 2003; Sherin, 2000). These studies outline a framework for understanding and nurturing students' "metarepresentational competence" (diSessa, 2004)—their capacities for such activities as creating, critiquing, comparing or quickly learning new representations. Armed with these insights, the critical question is not why students struggle to translate among standard representations or to perceive the function they simultaneously represent, but rather how or through what processes they come to construct meaning for and relations among those representations.

In a similar vein, Greeno and Hall (1997) assert that instruction in mathematics and science should emphasize not learners' familiarity with standard representational forms, but rather opportunities to engage in forms of representational practice. They argue for a perspective on representational artifacts as tools not only for solo reasoning about phenomena, but also for expressing ideas and communicating information in collective activity and social interactions.

Along these lines, a number of recent studies have demonstrated the importance and the utility of examining representations not as abstract objects, but as material elements of dynamic cultural practice and mediators of social interaction (e.g. Cobb, 2002; Danish & Enyedy, 2006; Hall, 1996; Hall, Stevens & Torralba, 2002; Latour & Woolgar, 1986; Meira, 1998). Representational artifacts serve as resources not only for solving problems, but also for coordinating interaction and interpretation (Pea, 1992; Roschelle, 1992; Woolgar, 1990), communicating ideas (Greeno & Hall, 1997; Pea, 1994) and orchestrating complex tasks across distributed networks of participants and material resources (Hutchins, 1995).

Focusing on the representational practices and competencies of learners as they struggle with expressing mathematical ideas and relationships in communicative situations exemplifies a different theoretical stance than presuming their increasingly accurate uptake of canonical representations and conventional interpretations with instruction. Such analytic emphasis also draws attention to the ways that meaning of diagrams, tables and other displays is established through socially-situated efforts to reason with and to interact through artifacts, rather than through apprehending self-evident semantic properties of those artifacts. Accordingly, some authors eschew the word ‘representation’ altogether when discussing graphs, charts, symbolic equations and other markings displayed on paper, computer screens and other media—instead referring to them as ‘inscriptions’—to emphasize their material rather than representational properties (Latour, 1990; Pea, 1993; Roth & McGinn, 1998). The meaning of an inscription is not a stable property of an abstract object, but rather evolves over time and in relation to emergent goals, problem-solving processes and interactions (Meira, 1995). When faced with a mathematical problem, an individual learner works to make sense of the situation by establishing the meaning of representations as tools for supporting his or her reasoning. In the same way, participants in a discourse create and interpret representations in order to establish their functions as tools for coordinating their social processes and collaborative reasoning.

This account of representational meaning as provisionally established through practices of reasoning in interaction—rather than being 'read off' as self-evident properties of inscriptions—entails a corresponding critique of how multiple linked representations come to 'mean'. Roth and McGinn thus argue that "any relationship between inscriptions…holds because of agreed upon practices, not because of an *a priori* ontological identity between two inscriptions" (1998, p. 42). Even the meanings of standard representational modes and the links between them are cultural artifacts produced through long histories of collective practice and re-established in each new use (Sfard, 2008). In order for different representations of a specific mathematical function to be meaningful in relation to one another, the *computer-mediated links* between displays of symbolic, graphical and tabular representations must be complemented by *interpretive links* that are established and maintained through a group's collective activity and/or constructed by individuals.

Of course, though learners may not apprehend their relations as such, the linked displays on a graphing calculator or computer do represent the same function in a pragmatic sense: when a user changes an algebraic expression, the graphical and tabular displays also change. As Kaput (1992, 1995) says, they are 'yoked'. That such representations are considered to be yoked is, however, not tied only to the production or interpretation of an inscription during a learning activity, but to a broader set of mathematical conventions, established over generations and now encapsulated in artifacts such as computational devices. Learners must be familiarized with those representational conventions so that they may participate in mathematical practices that presume the semantic mappings between the inscriptions across these different representational modes.

The intent of the learning environment designer—to make these mappings evident in the connected changing states of the inscriptions—is not directly perceivable by the learner, but will require interpretive work on the part of the learners and the teachers fostering the learning conversations around these representations (Roschelle, 1996). In recognition of these interpretive realities, we will continue to use the terms 'representations' and 'representational tools' rather than 'inscriptions' to foreground our goal that students learn conventional correspondences across different representational modes.

A focus on representational practices and interactions thus suggests shifting focus from the mathematical object common to a network of multiple linked representations, to the meaning for representational tools constructed or enacted through the reasoning processes of learners, and jointly negotiated through interactions among multiple participants in collective and collaborative processes. Our analysis accordingly seeks to examine the forms of reasoning and interaction through which learner groups interpret a set of dynamically linked function displays and other representational tools as they develop collaborative problem-solving strategies. In the next section we outline a design framework for supporting and investigating student collaboration with multiple representations, namely by situating linked representations of a common function in a distributed network of participants, devices and representational tools bound together by a shared task.

Linking Multiple and Distributed Representations in a Collaborative Task

The designed environment presented in this paper seeks to capitalize on the links among multiple representations in creating contexts for collaborative learning. Learners are provided with a variety of representational tools (described below) linked by a network of devices, and asked to use those tools together in order to develop strategies and actions for solving a collective task. Thompson (1994) argues for the design of learning environments that focus not on functions as abstract objects that learners may or may not construct through engagement with their multiple representations, but rather on aspects of specific situations which students might themselves be able to represent, and learn to recognize across different forms of representational activity. Kaput (1998) likewise stresses the importance of connecting linked function representations to multiple descriptions of real situations. The approach of this design study similarly seeks to support learners' efforts to interpret linked equations, graphs and tables by providing opportunities to make those displays meaningful as tools in relation to a specific problem context, in this case decrypting ciphertexts. In order to develop a framework for conceptualizing a system of several artifacts jointly engaged by multiple participants we will consider multiple linked representations in relation to a very different plural notion of 'representations': *distributed representations* coordinated in the accomplishment of a task.

Distributed representations are elements in a cognitive system that may include both internal representations associated with individuals in that system, and external representations (Zhang & Norman, 1994). While multiple linked representations are bound together by a common referent, distributed representations are united by their shared relevance to a common task. And while analyses of learning with multiple linked representations are generally oriented toward an individual learner who simultaneously engages those different representational modes, a distributed cognition perspective shifts our analytic attention to consideration of multiple participants as well as multiple representations. For example, in Hutchins' (1993, 1995) accounts of the coordinated activity of groups of people and tools guiding large ships, the computational work of navigation involves the propagation of representational states across an array of media linked to multiple actors. In order to determine the ship's location at regular intervals over the course of the journey, a team of navigators uses an array of tools to generate, translate, and reorganize a series of representations of their spatial relationship to known landmarks. A distributed account of the ways this team solves the problem of navigation involves expanding the unit of cognitive analysis beyond the mind of any individual in the system to include multiple social and material actors organized in relation to collective activity.

The learning environment presented in this paper distinctively integrates a set of multiple linked function representations with a system of distributed representations of a problem, and allocates access to and control of this array of representational resources among several participants in a collaborative task. The intent behind this design is to simultaneously capitalize on the potential for multiple dynamically linked representations to provide students with multiple views of mathematical relationships, and for a distributed array of representational tools to draw an array of learners together to make shared meaning of those representations in relation to collaborative activity.

Beyond providing a framework for analyzing the ways individuals work in concert with one another and with features of their environments toward the accomplishment of tasks, distributed cognition perspectives may also offer considerable utility for conceptualizing designs for collaborative learning (e.g., Stahl, 2006). Collaborative tasks are likely to be most effective when they are sufficiently open-ended and complex to necessitate contributions from each member of a group (Cohen, 1994), and when participants engage the task and one another in ways that sustain their diverse contributions (Barron, 2003)—in other words, when circumstances both allow for and encourage the assemblage of students to truly function as a group. The design presented in this paper thus seeks to employ a distributed network of participants and representational artifacts in problem-solving tasks that necessitate their joint interpretive efforts to turn those representational artifacts into their problem-solving tools.

Critiques of distributed cognition theory (e.g. Bereiter, 2002; Cobb, 1998) worry that extending accounts of the collective capacity of an ensemble of material and social actors into classroom settings may lead to an impoverished account of what we know about individual learners. And as Schwartz & Martin (2006) note, most work on distributed cognition to date has fallen short of attending to distributed learning—the extent to which distributed cognitive systems of individuals and their material or social environments adapt, change or reorganize themselves over time. Our intent in adopting a distributed perspective is to enrich rather than replace an account of individual learners' interpretive activity with representational tools. Our analysis below therefore seeks to characterize learning in terms of the dialectical relations between the adaptive reconfigurations of a group and the interpretive efforts of individual participants (cf. Cobb, 1999; Enyedy, 2003). This approach calls for a dual focus—on the distributed array of learners and displays as analytic unit for examining the development of strategies for solving decryption problems, and on the efforts of participants to establish individual and shared interpretations of representational tools that support those strategies.

To that end, this study addresses two central research questions:

1) How do the developing problem-solving strategies of groups draw on various elements in a distributed array of function and problem representations?
2) How do learners work together to establish ways of interpreting multiple and distributed representational tools as they develop and enact collaborative strategies?

THE *CODE BREAKER* LEARNING ENVIRONMENT

This paper draws on data from a classroom design experiment in which students used a variety of representational tools to solve collaborative problem-solving tasks. Those tasks formed the heart of a curricular unit titled "Code It!" and set in the applied context of cryptography[2]. This unit sought to introduce the topic of mathematical functions and support students' developing fluency with a variety of representational tools, as defined below. Students were asked to imagine themselves as cryptanalysts, and to collaborate with the other members of their small group on daily problem-solving activities as they attempted to break codes that the teacher or other student groups had created using algebraic expressions. The tasks were organized around uses of mathematical functions as substitution ciphers—codes that assigned each letter in a plaintext message to its ordinal alphabetic value between 1 and 26, and then mapped that value through a polynomial function to produce its corresponding output value in a numerical ciphertext alphabet. Groups were presented with a succession of messages that had been encrypted in this way, and charged with determining the unknown function ("breaking the code") used to encode each in order to decipher it.

Each student was equipped with a handheld computer running a software application called *Code Breaker*. That application provided students with what we will call *representational*

[2] Code It! was an adaptation of the "Codes Inc." curriculum unit originally developed at the Institute for Research in Learning by Goldman, Greeno and colleagues in MMAP (Middle School Math through Applications Project) to provide a slice of the real world chosen to engage young people in mathematics (Greeno et al., 1999), in this case having students role-play cryptologists designing and testing codes for clients. Their curriculum used code making and code breaking as a vehicle for teaching algebraic functions. For the Codes Inc. unit, students used software running on desktop PCs to create and break codes, and passed coded messages to each other via e-mail. When the second author wished to develop a classroom network employing handheld wireless computers for collaborative math learning, he sought out Goldman's collaboration to re-design the activities so as to support collaborating groups to compete with one another in creating and breaking codes (see Goldman, Pea & Maldonado, 2004; Goldman, Pea, Maldonado, Martin & White, 2004 for design rationale and preliminary accounts of this intervention). We thus adapted the Codes Inc. unit for small group collaborative learning using handheld wireless PocketPCs. In this new *Code Breaker* system, handheld devices communicated with each other via a classroom-based server computer, rather than directly, and different representations of candidate functions were updated dynamically on the handheld displays of each group member, unlike the original Codes Inc. design.

tools—an array of different displays featuring information about the encrypted text, and a set of dynamically linked representations using different representational modes of a candidate solution function generated by a student group using the application. The static ciphertext displays together made up a distributed array of representations of the problem: each highlighted different characteristics of the encrypted message and the numerical characters it comprised, and thus provided different potential clues about the unknown encoding function. The linked displays representing the function formed a set of multiple representations of a candidate solution: they showed algebraic, graphical and tabular views of a given mapping between the plaintext alphabet and a corresponding set of possible ciphertext values. The problem-solving process for learner groups involved seeking ways to align these different problem and solution displays so as to make inferences about the parameters of the unknown encoding function for the ciphertext. Students used the *Code Breaker* tools to generate and to examine candidate encoding functions, and to compare the displays of each of these functions with the resulting characters in the ciphertext message to match one such candidate to the encoding function.

A distinctive aspect of the socio-technical design of the collaborative *Code Breaker* software employed in this study was that the handheld devices were linked through a local wireless network such that students who were seated together in groups of four were assigned to a corresponding server-defined group (Figure 1). In consequence, changes to a function that was displayed on one student's handheld automatically propagated to the devices used by members of that group. Though a single device could display only one or two of those linked representations at a time, a group could collectively examine the full array simultaneously by collectively interpreting the representations that were displayed on their set of devices. The activity design thus takes the form of a multiple representations 'jigsaw' (Aronson, 1978;

Cleaves, 2008); each group member was assigned responsibility for viewing one or two representations, and these responsibilities were rotated daily, with the intent of each student developing facility with each representational tool and a deeper understanding of its distinctive affordances for decrypting the ciphertext. In principle, each student could take any view; all the representations were stacked in a long viewing window through which students scrolled to their assigned function representational mode (e.g. graph, table). Students could and sometimes did view representations other than those assigned to them; these departures were in keeping with the open-ended nature of the task. The purpose of the assigned roles was not to rigidly structure the problem-solving process around a precise arrangement of representations, but rather to create a flexible network of social and technical linkages through which students could learn to select, combine, and coordinate representations as resources for analyzing functions and breaking codes. Whether this jigsaw-like role rotation would be reflected in how the learners organized their activities as a distributed reasoning system remained to be determined empirically.

Representational Tools

The *Code Breaker* software included four different dynamically linked representations of the candidate solution function: 1) an algebraic equation, 2) a graph, and two different tabular views: 3) the function table, and 4) the inverse function table. Three other representational tools, the 5) ciphertext table, 6) frequency table, and 7) word frequency table, displayed static features of the encrypted message. In addition, two of these dynamic function displays, the graph and the inverse function table, also included information about the encoded ciphertext message. This overlap between some problem and solution representations was intended to support students' efforts to interpret candidate function displays in relation to the encrypted message, and to highlight properties of the unknown encoding function. Figure 2 illustrates the relations among the different *Code Breaker* representational tools—those that simultaneously displayed dynamically linked representations of the candidate function, those comprising a distributed system of representations of the encrypted message, and those that integrated problem and solution representations. The arrows in this figure indicate dependencies among these different displays, so that the curve displayed in the graph and the dynamic values in the function tables were generated from the algebraic candidate equation, while the characters and counts in the frequency tables, the vertical window dimensions of the graph, and the static values in the inverse function table were all determined by the original plaintext message and the encoding function which together produced the ciphertext. These various candidate solution function and ciphertext displays are each described below.

[Insert Figure 2 about here]

1. The Candidate Function Equation. One student in each group was assigned responsibility for viewing and manipulating a symbolic expression for the candidate function. Encoding and candidate functions were of the form $y=ax^b+c$, where *c* could be any integer, *a* any nonzero integer, and *b* could equal one, two, or three.[3] Candidate parameters were adjusted from their default settings (*a*=1, *b*=1, *c*=0) by tapping with a stylus on either the top or the bottom half of the number on the handheld screen, causing the value to increment or decrement by one unit. In order to break a code, groups needed to correctly determine these three parameters of an unknown encoding function and enter them in the candidate display.

2. The Graph. The *Code Breaker* graph (Figure 2) displayed the candidate function curve in a window scaled in accordance with the encoding function. The x-axis of the graphing window

[3] In addition to the polynomial mapping, some codes also featured an "offset." Offsets shifted the relationship between the letters of the alphabet and their ordinal values, so that while an offset of zero meant that A was associated with an input value of one, an offset of one associated A with an input of 2, B with 3, and so on, including associating Z with 1. Codes featuring offsets were introduced approximately halfway through the three-week series of decryption activities, as part of an effort to steadily increase the difficulty of the tasks.

was always fixed to the alphabetic domain, spanning from zero to twenty-six. The y-axis, on the other hand, adjusted automatically around the range of values included in the ciphertext message. Each of those ciphertext values was also represented in the graph by a horizontal line stretching from the y-axis until it intersects the candidate curve. A corresponding vertical line, drawn from this intersection to the x-axis, reflected the mapping of the ciphertext value through the inverse of the candidate function.

3. The Function Table. At different stages of the study, the *Code Breaker* software featured two different table displays dynamically linked to the candidate function. The function table (Figure 2) paired a static X-column, displaying the numbers one through twenty-six, with a dynamic Y-column that updated with each adjustment of the candidate function to show the new output value to which that polynomial function would map each of the X-values.

4. The Inverse Function Table. The inverse function table mapped the range of ciphertext values in an encrypted message, shown in the Y-column of Figure 3a, through the inverse of the current candidate function (y=17x-29), with the result displayed in the corresponding cell of the X-column. When this process yielded an integer from one to twenty-six, the corresponding plaintext letter appeared in the "Letter" column.

The version of *Code Breaker* implemented at the outset of this study featured an overlooked software bug with significant implications for the behavior of the inverse function table, and for the cases presented below[4]. This tool contained an additional feature intended to round to integers values that were nearly but not exactly integral in order to signal to students that they were getting close to a letter match for cases of quadratic and cubic functions with large parameter values. In fact, the bug caused the table to include *all* non-integer values with a

[4] For a more detailed recounting of the inverse function table, its bug, the insights it occasioned, and the reasons for its replacement, see Author (2008).

greatest integral part between 1 and 26. In linear cases where the coefficient of the candidate function was greater than that of the encoding function, the table displayed multiple ciphertext values mapped to the same letter. Figure 3b reveals what the 'buggy' table actually displayed, given the code and candidate functions from Figure 3a. After the discovery of this bug during the first week of decryption activities, the inverse function table was replaced by the function table.

[Insert Figures 3a and b about here]

5. The Coded Text. The first of the static problem representations (i.e., those which are unchanging even as the candidate solution functions are changing) featured the ciphertext message itself[5]. This display showed a series of numbers comprising the encoded letters of the encrypted message, with spaces between numbers bounding each letter and brackets around strings of numbers to demarcate words (Figure 2).

6. The Character Frequency Table. The Character Frequency Table was composed of three columns. The Y-column displayed all the numbers that appeared in the ciphertext of the currently encrypted message. The "Count" column specified the number of times that each of those ciphertext values appeared in the coded message. As with the function and inverse function tables, these columns were divided in half, with the two halves placed side-by-side to save screen space. The table was sorted according to the count, from least to most frequent, rather than according to the relative numerical values of the ciphertext characters in the Y-column[6].

7. The Word Frequency Table. Similarly, the word frequency table displayed the number of occurrences of each word in the ciphertext (Figure 2). As in the Coded Text, the words in the encrypted message were displayed as strings of numbers, with spaces around each distinct

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[5] Hereafter, we use "Coded Text" to refer to this representation in the *Code Breaker* software, and "ciphertext" to refer to the actual encrypted text message or the set of numerical characters it includes.

[6] The X-column shown in figure 6 was added after the removal of the buggy inverse function table, and retained some of the same functionality, displaying a value whenever the inverse of the current candidate mapped a ciphertext value to an integer between 1 and 26.

number to identify letters and brackets around each sequence of numbers that comprises an encoded word. Each such word was listed singly on a line of the “Word” column in the table, and the number of times that word appeared in the message was displayed to the right in the “Count” column.

Decryption Strategies

Strategies for breaking codes generally involved establishing interpretive links between candidate function and problem representations. Those links might be made in either direction—from the variation of candidate function parameters to the examination of the resulting curves or values in comparison to those in the ciphertext, or from examination of ciphertext-based displays to inferences about possible candidate function parameters. The first path involves using the linked function representations to examine how a given candidate maps the plaintext alphabet in relation to the set of ciphertext values—how well the graph ‘fits’ the adjusted window by spanning the domain and range specified by the plain and ciphertext sets respectively, or how the function table values compare to those in a ciphertext display.

A second strategic approach might begin with the values in the ciphertext, and involve matching one or more of these with a corresponding plaintext letter to form an ordered pair. The intent behind the *Code Breake*r design was for student groups to use approaches of each kind iteratively and in combination. Doing so follows from the collaborative context and the networking of multiple devices, as different students in the group can be simultaneously engaged with different linked representational tools. Moreover, while these codes could in principle be broken using only analyses of the ciphertext displays, the candidate function representations were quite useful not only for determining the degree and often also the lead coefficient of the polynomial encoding function, but also as scaffolds for students who did not yet have the familiarity with functions or the fluency with algebra to easily or successfully enact such analyses.

METHOD

Context and Participants

The *Code Breaker* handheld environment was implemented with two middle school mathematics classes during a five-week summer school session during which students had lessons introducing the topic of functions, the evaluation of simple polynomial expressions, coordinate graphing, exponents, and function tables. Students were also introduced in the first two weeks to the history and key terminology of cryptography, the principles of substitution ciphers, and the mechanics of the handheld computer user interface. Students spent the rest of their time over the remaining three weeks collaborating in small groups to make and break codes. As they grew familiar with the different *Code Breaker* representational tools, students in each group were encouraged to work together to develop their own strategies for using those resources to solve increasingly challenging decryption problems.

The focus of the summer program was on students' preparation for introductory pre-algebra and algebra courses in the school year, with emphasis on their familiarity with polynomial terms and expressions, with coordinate graphs, and with numerical patterns and relationships. The integration of *Code Breaker* activities was intended to support students' developing familiarity with a variety of representational resources as tools for examining functional relationships and solving problems. The learning environment aimed to present students with opportunities for reasoning with and making connections across different function representations, with an eye toward developing their fluency in interpreting and reasoning with representational tools, analyzing numerical patterns in the ciphertext set, and investigating functional relationships represented by the codes. The extended analogy between codes and functions provided a context throughout the instructional unit for supporting students' learning about both the relations between specific input and output values of a function—as generated by an algebraic rule, expressed numerically, and located graphically (as Cartesian ordered pairs), and the relations between input and output sets of values—the domain and range of the encoding function. A more focused examination of opportunities for learning about functions in this environment and through this extended analogy with cryptography is presented elsewhere (White, 2009). Briefly, that analysis found that groups' different decryption strategies emphasized different aspects of function. While structural properties of functions were particularly salient in some strategies, other strategies relied on operational features of functions, and other approaches integrated elements of functions as both processes and objects; groups were able to determine encoding function parameters more precisely and break more challenging codes as they developed decryption strategies that reflected these alternative perspectives on functions. More detailed accounts of individual student learning outcomes in relation to the instructional unit are likewise reported elsewhere (White, 2005; 2006). The specific contribution of the present paper involves examining students' activities with and developing interpretations of representational tools, particularly as resources for examining numerical patterns and relationships, and for solving problems collaboratively.

The student population was highly diverse in terms of prior academic achievement. While some students were required to attend the summer program due to poor math performance during the prior school year, others were high-attaining students voluntarily participating in the course for enrichment. Collaborative groups were organized to reflect that diversity in prior achievement; student assignments to their groups were based partly on a pretest score[7], so that each group included students at a wide range of performance levels on that instrument, and partly on the observations of the teacher and researchers. Whenever possible, groups remained the same for the duration of the study.

This paper analyzes the problem-solving work of two groups, comprised of four students each. A total of four groups from the two classes were selected to be videotaped during all small-group activities for the duration of the study. These four groups were purposively selected according to several criteria, including the consent of all members to be videotaped and interviewed, and informal observations of their work during preliminary activities. Groups One and Two were selected from among the four focus groups for more detailed analysis because they were more often on-task than one of the remaining groups, and because their use of a wide variety of representational resources and decryption strategies during decoding activities was more readily observable from the video record than that of the other. Of those other two groups who were regularly videotaped, one featured a combination of absences and off-task disagreements and consequently struggled to work to collaboratively or to develop effective ways of using the *Code Breaker* tools. The fourth group performed similarly to Groups One and Two in terms of engaging in the activities and developing successful decryption strategies over the course of the unit, but did not discuss, debate or narrate their specific uses of different representational tools as often or as audibly. In contrast, each of Groups One and Two was clearly observed by both researchers and the teacher coming up with novel strategies or ways of using representational tools over the course of the unit and thus identified as being of particular interest for subsequent analysis. Thus we do not claim that the groups analyzed below were

[7] Topics on the pretest and an identical posttest included reading graphs and tables, evaluating arithmetic and simple algebraic expressions, solving linear equations, and vocabulary words related to functions, algebraic expressions and graphs. Focus group students' results on these assessments are reported in White (2006).

typical of the larger sample, though their overall success in decryption activities and performance other classroom tasks was not exceptional. They were selected for their relative tendencies not only to work, but also to discuss that work with one another—thus providing the researchers with a wider window into their problem-solving processes than that of other groups. In this sense, we take them as exemplars of the kinds of activity by students at this grade level that might be possible to support in this learning environment, rather than indicative of the overall success with this iteration of the design in bringing about that kind of activity across all groups.

Because gender was not a factor by which students were assigned to groups, there were some same and some mixed-gender groups among both the classes at large and the six focus groups; Group One was composed of four girls and Group Two of four boys. The varying levels of prior academic achievement among the students in each group are partly illustrated by their performances on the pretest. In Group One, Tina had the highest score in the class and Jessica both scored above the upper quartile, while Shirley and Monique both scored below the first quartile. In Group Two, CJ scored above the upper quartile, Vince above the median, and Reggie and Jason above the lower quartile. Tina was a rising 8^{th} grader who had studied Algebra I during the previous year, while all other students in the two focus groups were rising 7^{th} graders who had received regular 6^{th} grade math instruction the prior year.

Analytic Approach

To characterize groups' developing decryption strategies, and the use of representational tools on which those strategies relied, we examined all 32 problem-solving episodes undertaken by the two focus groups over the study's final three weeks. These 2 to 30 minute decryption events began when a group downloaded a new code to break, and concluded when they either solved or stopped working on the code. Video records of each event were transcribed and reviewed in detail. When applicable, server logs of student decoding activity, researchers' notes, and students' written records were also consulted to supplement the video data.

Each decryption event was analyzed with regard to the ways students attempted to coordinate participants and representational tools in order to draw inferences about the solution function, what we call decryption strategies. In particular, we reconstructed the problem-solving processes enacted by each group in each event in terms of 1) the different displays examined by each participant, 2) their efforts to establish interpretive links between those displays, and 3) their efforts to determine parameter values of the unknown encoding function. We represented these group processes for each decryption event as a particular configuration of tools and participants corresponding to a subset of the representational tools and linkages depicted in Figure 2. In other words, the analysis of each decryption event included a diagram of the candidate function and ciphertext displays from which students sought to derive information about the encrypted message and inferences about the encoding function. However, whereas Figure 2 uses solid arrows to represent dependencies between these displays, the analytic diagrams use dashed arrows to represent instances in which students made inferences about candidate function parameters based on information in the specified representational tool(s) (see, for example, Figure 11 below).

These depictions of decryption strategies provided a means to examine learning in this environment at the level of the distributed group, particularly as emergent and changing arrangements of participants and tools in relation to successive tasks. These different distributions of students and displays were then examined in terms of the ways they were supported by, and supportive of, students' individual and collaborative efforts to reason with representational tools. In particular, we sought to identify instances in which groups adapted into new alignments, and began to implement new strategies, especially those that they would go on to use repeatedly and with success. Below, we present several episodes that illustrate key problem-solving approaches and configurations of representations used by these two groups, and the moments and processes through which they emerged in each group's work. In each case, we examine the ways that students worked to coordinate interpretations of *Code Breaker* displays, and to establish interpretive links between displays.

Because students could and often did freely adjust the scroll position of the viewing window on their devices, the available data did not always provide a complete picture of the different displays students examined as they worked to complete decryption tasks. Students often made explicit reference to the display they viewed, or simply adhered to the view associated with their role assignment. However, there were also instances in which we relied on close scrutiny of the video record in order to track students' scrolling adjustments in relation to the timing of a relevant observation, action or inference. In other instances, students' views could be accurately reconstructed based on their reports of information that was only available in a particular view. Instances in which the particular display one or more students sought to interpret or use in a given set of utterances or actions could not be clearly inferred through these means were excluded from the analysis.

RESULTS

Tables 1 and 2 summarize the patterns among successive problem-solving strategies for each group. Over the course of three weeks, each group gradually moved from strategies for determining encoding function parameters that relied only on inferences from linked representations of the candidate function to those that involved establishing and capitalizing on links between representations of both the candidate function and the encrypted text. The functions used to encode these messages generally grew more complex as each group progressed through the tasks, featuring gradual increases from first to third degree polynomial functions, and the eventual introduction of codes with offsets[8]. Group One's first successful code breaking effort came in their third task; after that, they solved all but two of the tasks they undertook even as those grew increasingly difficult, including every task in which their decryption strategy included establishing interpretive links between representations of the candidate function and the ciphertext. Group Two successfully solved several codes based on relatively simple linear functions in Tasks 5-8 using only inferences from candidate function representations, but struggled thereafter until they developed an approach to linking candidate and ciphertext representations over their final three tasks.

[Insert Tables 1 and 2 about here]

Below, we present a series of episodes to illustrate the ways the two groups moved from strategies that relied exclusively on inferences from dynamically linked function representations to those in which they established interpretive links between candidate function and ciphertext representations. We examine the ways these changing distributions of participants and displays were enacted through students' efforts to reason and interact with multiple and distributed representational tools. In episodes One through Four, we present a sequence of several events that highlight the interpretations of and the interactions with different displays through which Group One moved from problem-solving approaches that relied exclusively on candidate function representations and their display features, to ones that integrated ciphertext displays. In Episodes Five through Seven, we illustrate the strategic development of Group Two somewhat

[8] The groups each worked on some messages assigned by the teacher, and some encrypted by other groups within parameters established by the teacher for a given session (e.g. only linear encoding functions). Consequently, the respective series of codes attempted by the two groups were similar, though not identical.

more succinctly, and with an emphasis on contrasting features as they came to develop quite different ways of using these representational tools to break codes.

Episode One

This episode finds Group One engaged in their second day of decryption activities with the *Code Breaker* tools, and illustrates their early efforts to interpret and apply this array of representational resources. The code they sought to break had been encrypted with the function y=4x+7. Tina was responsible for editing the candidate function in this session, Jessica for examining the Inverse Function Table, and Shirley the Graph. Because of its proximity to the candidate function in the scrolling window, Tina could also see the graph on her own device. Monique, who was assigned the Frequency Tables, had been busily writing in the group's logbook just prior to the start of the episode, and was still in the process of adjusting her display to that view as Tina, Jessica and Shirley began to coordinate a series of actions and interpretations around the linked Equation, Table, and Graph. As Tina edited candidate function parameters, the students reacted to the changing graphical and tabular displays. Our focus in analyzing this excerpt is on how these students coordinated these and other representational tools and interpreted the links among them in relation to the decryption task.

1. Jessica: [Watching changes in the IFT as Tina increases candidate coefficient from 1 to 2] Oh, keep on doing it! I got more letters. Keep on going. Go, go…Go.
2. Shirley: [Looking at the Graph] Now it looks…now we're looking…
3. Jessica: Oh! Keep on going, yeah!
4. Tina: Wait, wait. [increases coefficient again] Do that, do that, do that?
5. Shirley: Yeah. Keep on going.
6. Jessica: No, go, go, yeah. Keep on going.
7. Shirley: More.
8. Jessica: Keep on going.
9. Shirley: More. More, more, more.
10. Jessica: OK, that's enough.
11. Shirley: No, more.
12. Tina: [increases coefficient from 4 to 5 and looks at the Graph] OK, that's a little too much.
13. Jessica: Go back.
14. Shirley: You want to change another one? Change the constant?

15. Jessica: Maybe you should try changing the constant.
16. Tina: [gasps] Hey we did it! Four x plus seven.

Summary. Jessica's instructions were based on her observation of the increasing number of different letters appearing in the Inverse Function Table as Tina adjusted the equation. Candidates with linear coefficient three and four spanned more of the full alphabetic domain over the set of ciphertext outputs than those with coefficients one or two, and so indeed caused more letters to appear as Tina unitarily increased that coefficient from one to four (Figures 5a-d). Meanwhile, Shirley gave her instructions based on a graphical version of the same information, as Tina's adjustments to the candidate coefficient brought the graph into a steadily better fit with the window dimensions spanned by the alphabet on the x-axis and the ciphertext values on the y-axis (Figures 4a-d).

[Insert Figures 4a-d and 5a-d about here]

Analysis. These three students successfully coordinated their interpretations of three linked candidate function representations to determine parameters of the encoding function. They did not, however, consider distributed representations of the underlying problem situation apart from those aspects embedded in candidate function display features. Figure 6 illustrates the arrangement of students and representational tools through which the group coordinated their interpretations of the decryption problem across several displays with their actions on the candidate function Equation. Their inferences about that Equation relied on the Graph and Inverse Function Table as resources for comparing representations of the candidate function with information about the encrypted message incorporated into the displays—the vertical dimension of the graphing window, and the ordered lists of ciphertext values and possible associated letters in the Inverse Function Table. These inferences were retrospective; rather than anticipating precise parameter values based on examination of the ciphertext, they simply evaluated whether to “keep on going” after each adjustment. Jessica’s use of the Inverse Function Table in this way would prove to be limited in scope, and the tool itself was soon removed due to the discovery of the bug. By contrast, this emerging approach to using the graph would figure into their problem solving efforts in each of the remaining 12 tasks they undertook. However, neither Graph nor Inverse Function Table provided sufficient information about the encrypted text and the encoding function to reliably solve more challenging codes, and despite their success in this episode the group would soon seek other ways to make use of additional representational tools. Indeed, while they determined a precise value for the candidate coefficient by comparing values of 4 and 5 using both displays, their subsequent success in finding the correct constant value was nearly accidental; having set the coefficient, Tina simply incremented the constant until the plaintext display showed a precise match. They would not be so lucky in later tasks; the next episodes detail their development of more deliberate approaches to reasoning about ciphertext values and precisely specifying encoding function parameters.

[Insert Figure 6 about here]

Episode Two

Over the following three episodes, we highlight the ways the group began to reorganize itself into a new configuration of participants and representational tools. In particular, our analysis will illustrate the ways this adaptation at the level of the distributed group depended on and followed from the efforts of individual participants to make sense of, and to coordinate their interpretations of, several representations of the ciphertext. The next episode took place on the same day as and immediately following Episode One, and finds Group One in the midst of an effort to decode a message that had been encrypted using the function $y=-4x+7$, and so contained predominantly negative values. As in the previous episode, Tina edited candidate function parameters, and she and Shirley noted changes in the graph while Jessica did the same in the inverse function table. This time, however, that arrangement of students and representations proved less fruitful. After proceeding this way for about three minutes, Tina had adjusted the coefficient to negative three based on observations of the graph (see Figure 7), but they were unable to progress further using only the linked representations of the candidate function.

[Insert Figure 7 about here]

Instead, the group began to turn their attention to other representational tools in search of additional information about the Coded Text. The message they were trying to break did not contain the letter Z, so the extreme values of its range were 3 and -93, respectively representing A and Y. In the segment that follows, Tina asked Monique to report the lowest value in the ciphertext based on her view of the Word Frequency Table, prompting a series of several responses from several different representational tools:

17. Tina: What's the highest…number in the word frequency table? I mean, what's the lowest?
18. Monique: [Looking at the Word Frequency Table] Umm…three.
19. Tina: [Looks at the Coded Text] That's the highest. Cause they're all negative.
20. Monique: Oh yeah, so positive. OK.
21. Jessica: [Looking at the Inverse Function Table] It’s forty-one.
22. Tina: [Leans forward to look closely at the Coded Text] I'm looking at negative seventy-seven.
23. Monique: The lowest… Negative twenty-five.
24. Jessica: Oh. No, I got…well, never mind.
25. Monique: No, negative one.

Summary. In this brief segment, these three students examining three different representational tools produced five incorrect candidates (3, -41, -77, -25, and -1) for the lowest value in the ciphertext. Both Tina’s difficulty in formulating her question (line 17) and Monique’s struggles to answer it correctly illustrate their efforts to make sense of the meanings of “highest” and “lowest” in the context of a set of numbers that were predominantly negative, and of displays that did not order those numbers in terms of relative value or otherwise make extreme values salient. Monique’s successive proposals for the lowest number (lines 18, 23, 25) likely reflect each of these challenges. The word frequency table presents ciphertext characters in relation to their position in words in the encrypted message and the respective frequency of those words, rather than in order of their relative value. Thus, -1, the ciphertext value with lowest absolute value, appeared only once in the Word Frequency Table, while 3, the next lowest absolute value, appeared seven times (Figure 8a). After Tina asserted in line 19 that 3 was "the highest" because the other values were "all negative", Jessica next offered -41 based on her view of the inverse function table (line 21). Again, this incorrect report may have reflected Jessica's interactions with the features of the representation rather than her understanding of relative value. The inverse function table divides the range of values in the ciphertext into two columns. Thus it appears that Jessica simply read from the bottom of the wrong column, where she would have seen -41 rather than the -93 at the bottom of the next column (Figure 8b). In offering -25 and then -1, Monique appears to have been converging on successively lower absolute rather than relative values as she continued to scan the Word Frequency Table (lines 23, 25). And Tina's own proposal of -77 (line 22) may have been the lowest she saw in the Coded Text, where it appeared in the first word and where she would have had to scan to the bottom row to see -85 or click the down arrow in the lower right corner of the ciphertext display in order scroll toward the further down in the passage and see -93 (Figure 8c).

[Insert Figures 8a-c about here]

In the moments that immediately followed, they would resolve the misunderstanding about relative values, as all four students in the group came to agree that -77 was the "lowest" value among their respective candidates:

26. Jessica: [Looks up at Monique] No the, the…
27. Shirley: [Looks up at Monique] No, no, that's not the…
28. Jessica: The higher the negative, the, the lower…the positive.
29. Tina: [Looking at Candidate Equation and Graph] So this has got to be it. [Begins editing the constant]

30. Monique: [After looking up in response to Jessica and Shirley's comments, turns back down to look closely at the Word Frequency Table] Oh then it's, then it's negative…seventy-seven.
31. Jessica: I got…
32. Shirley: Seventy-seven? [looks back down at Graph]
33. Monique: Yeah.
34. Shirley: [Watching Graph as Tina edits candidate function] OK, it's going somewhere…
35. Monique: [Looks up from her PDA, sits back, speaks softly] I just learned something. Cool.

Summary. As Monique settled on -1 as the lowest number in the code, Jessica and Shirley both recognized her confusion and moved quickly to correct her (lines 26-28). Though imprecise, Jessica's explanation that "higher" negative values would correspond to "lower" positive ones resonated; on hearing it, Monique turned her attention back to the word frequency table, scanning for a few seconds before correctly identifying -77 as the lowest of the reported values (line 30). Moreover, she appeared sufficiently confident about her interpretation of this number as the lowest value that she affirmed her report—after studying the word frequency table for a few moments more—when Shirley questioned it (lines 32-33). As the rest of the group continued analyzing the code, Monique leaned back in her chair, and addressed no one in particular as she announced that she had "just learned something" (line 35). Meanwhile, Tina had already decided, based on the presumed low value of -77 and the appearance of the graph, that the candidate coefficient of -3 had "got to be it" (line 29) and begun decrementing the constant.

Analysis. Figure 9 depicts the new arrangement of students and representational tools emerging in this episode. While the group continued to draw inferences about the candidate function parameters only from the Graph and Inverse Function Table, they also began to show considerable interest in and develop shared ways of seeking information in two representations of the encrypted message. As the next episode will show, these efforts to interpret a wider array of distributed problem representations would prove pivotal in their subsequent development of approaches to reasoning about encoding function parameters. Moreover, their attention to the extreme values in the ciphertext range would emerge as a central feature of their engagement with both the task and the underlying mathematics.

[Insert Figure 9 about here]

The current episode also illustrates that the process through which the group determined how to answer a question like the one they discussed here was not simply dependent on the relative affordances of these different representational tools or the mathematical knowledge of the students. Rather, their efforts to interpret and reason from code features like extreme values emerged from negotiations among students, representations, and the distinctive characteristics of a particular problem. For example, as this episode clearly illustrates, the Word Frequency Table was hardly well suited to finding extreme values. However, the group's joint efforts here to locate and to coordinate their interpretations of those values in that display would figure centrally in emerging regularities by which the group would both seek out characters representing one-letter-words in the Word Frequency Table, and associate those ciphertext characters with plaintext letters using interpretations of extreme values. Negotiations like these, we argue, were exactly the processes whereby the group learned to use these representations as tools for breaking codes, gradually coming to consensus around particular ways, and not others, of coordinating a distributed network of students and representations. And these collective efforts at strategic coordination of representations and participants, as this episode illustrates, were also opportunities for individual mathematical meaning-making of the kind displayed here by Monique. This instance in particular suggests an important mechanism by which the changing strategic configurations of the group might yield corresponding learning opportunities for its individual members, as in this instance Monique's understanding of what "highest" meant in relation to negative values was of importance to the group's reliance on her representational vantage point, and Jessica's and Shirley's efforts to coordinate that understanding would be crucial to the group's ability to function effectively in that alignment.

Episode Three

The next episode found the group engaged in additional efforts to coordinate individuals' views of different code representations in order to identify and interpret extreme values in the ciphertext. It also illustrates how they would begin to use those interpretations in combination with the identification of single-letter words in order to draw inferences about the encrypted message and the encoding function. This episode took place a day after the previous one, and the students had rotated role assignments so that Shirley was now editing the candidate equation, Tina examining the Graph, Monique viewing the Inverse Function and Jessica the Frequency Tables. The message they sought to break in this task was encoded using the function y = -6x + 14, and again included the plaintext letters A and Y but not Z. It also featured a single-letter word, which Jessica located less than a minute after the group began work on the task:

36. Jessica: [Looking at the WFT] I found a one-letter word.
37. Tina: You did?
38. Jessica: Negative forty, and there's… it happens two times in the poem.
39. Tina: OK. Tell me what's the highest number?
40. Jessica: That would be a positive, right?
41. Tina: Yeah. Highest positive number.
42. Jessica: There is…
43. Tina: [Looking at the Coded Text] Eight. I see eight.
44. Jessica: Eight.
45. Tina: Now what's the lowest negative number?
46. Jessica: Negative?
47. Tina: Yeah.
48. Jessica: I think that would…
49. Tina: [Looking at the Coded Text] I'm…I'm looking at negative one hundred and thirty six.
50. Jessica: Where's that?
51. Tina: On the top. Do you see it? Negative one thirty six?
52. Jessica: [Continues looking at the WFT] The…no.
53. Tina: OK. What was the, what was the negative number you had?
54. Jessica: [Still looking at the WFT] Oh, I see it. OK.
55. Tina: What was the number you had for the one-letter word?
56. Jessica: Negative forty?
57. Tina: Negative forty. [Begins scrolling down to the WFT]

58. Jessica: And…and you're wrong, cause um, the lowest, the highest number…wait, no. You're right.
59. Tina: [Looking at the WFT] Negative forty…[scrolls up to look at the graph]…and due to the graph that's the same I would say that negative forty would be I. So how do we get I to equal negative forty?

Summary. This exchange finds the group communicating with more mathematical precision, and coordinating representations more efficiently, than in Episode Two. Whereas in the previous episode three students used three different representations to derive several different and incorrect minima, here Jessica and Tina quickly and accurately found the same maximum value in the Word Frequency Table and the Coded Text (lines 39-44). Reaching agreement about the minimum value across these different representational views took more work, as Jessica had to scan for several seconds to find Tina's proposed value of -136, and briefly began to dispute the relative value comparison before agreeing it was the lowest (lines 45-58). Their eventual concurrence, however, reflects the effectiveness with which the girls were coming to use their respective representational tools in concert to determine these range values.

[Insert Figure 10 about here]

Having established the extreme values of the ciphertext, Tina sought to interpret other values in terms of the range between them, asserting that the one-letter word Jessica had identified earlier in the segment corresponded to the plaintext letter I (line 59). In attributing this correspondence "to the graph that's the same," she likely took advantage of the fact that the y-axis of the Graph, automatically scaled for this set of ciphertext values, would have been marked by tens from 10 at the top to -140 at the bottom (Figure 10). Because in this instance the one-letter word, -40, happened to be a multiple of ten, it appeared as a labeled scale mark on the y-axis. Moreover, Tina interpreted the positioning of -40 above the middle of the y-axis in the graphing window, and so somewhat above the middle of a span of numbers ranging from somewhat lower than -136 to somewhat higher than 8, as corresponding to the alphabetic positioning of the letter I. She also appears to have linked this graphical interpretation more directly with the low and high values in the range, as she demonstrated in explaining her rationale for linking -40 with I to her group mates a few minutes later:

60. Tina: OK…[picking up a pencil] there are letters, you know how they're turned into stuff, like A [writes an "A" on an open notebook page on her desk], through Z [draws a down arrow below the A, and then writes a "Z" below that]?
61. Monique: [nods]
62. Tina: You know, like, [gestures toward PDA] according to this graph, A would be the greatest number.
63. Monique: Um-hm.
64. Tina: [continues writing as she speaks, recording the values she references] And Z would be the least. The least number was one hundred...negative one hundred and thirty-six. And the greatest number was eight. So A probably is eight. But it's, [points to Jessica] she found that the one-letter word was negative forty. Now where would we put that in the range?
65. Jessica: In the middle.
66. Tina: Yeah, somewhere in the middle.
67. Monique: [puts her head down against her desk briefly] I have no idea.
68. Tina: [Looking at Monique, she holds both hands flat in the same plane, waving them first upward, then forward over her paper, as if along the arrow she drew earlier between A and Z.] But sort of in, a little bit more to, closer to the A than the direct middle [presses her index finger against the paper to indicate the letter A]
69. Monique: [slumps back in her chair and puts her hand to her forehead as Tina points to her paper. At this point, the teacher interrupts the group briefly]
70. Tina: [continuing with her explanation 90 seconds later] See, if you look at this [reaches across table to Monique's PDA, now scrolled to the top of the display, and points at the Coded Text with her stylus, indicating an 8], the greatest number is eight.
71. Monique: [hunching forward and leaning across her desk to look at the screen of her PDA, now between her desk and Tina's, as Tina points there] Um-hm.
72. Tina: And according to this [moves her stylus down to indicate the Graph], this means that A [points to A on the x-axis] will be the greatest number [points to the section of line displayed above the A] and Z will be the very very very least [follows slope of line down to location above Z]. Don't you think?
73. Monique: [reaches forward and taps PDA screen] Yes.
74. Tina: So. That's what happens when we do this. When we counted all these numbers, A, eight, was the greatest number, right?
75. Monique: [nods]
76. Tina: So that means A will probably be eight, right?
77. Monique: [sits back and looks up, pauses for two seconds] Yeah, because A is the greatest letter and eight is the greatest number.

Summary. This segment again finds the group engaged in careful efforts to coordinate their interpretations of information about the code distributed across several representational tools. Monique, in particular, was confused about how Tina knew that -40, the one-letter word in the message, was an I rather than an A, the other commonly occurring English single-letter word.

Tina's explanation, which began with a series of inscriptions on paper (line 60), briefly referenced the Graph (line 62), continued with pencil and paper (line 64), recalled a value found by Jessica in the Word Frequency Table (line 64), next linked a sequence of gestures to the paper-and-pencil inscriptions (line 68), and finally moved to the Coded Text (line 70) and then the Graph (line 72) on Monique's device. Across these several tools and inscriptions, Tina built an account of the mapping between the ordered letters A through Z in the plaintext alphabet and the ciphertext values 8 through -136, and used that mapping to link the relative positions of I among the plaintext letters and -40 among the ciphertext numbers. Though Monique appeared both confused and frustrated at some points in Tina's paper-based explanation (lines 67, 69), she engaged more fully as they came together to look at the same device screen (lines 71-75), and managed to partially articulate this interpretation herself by the end of the segment (line 77).

Analysis. In the first of these instances, Tina and Jessica worked to establish a shared interpretation of ciphertext values across different displays. And in the second, Tina likewise sought to articulate links between several displays and inscriptions in order to coordinate her interpretation of a ciphertext character with Monique's. None of these efforts involved simultaneous consideration of multiple candidate function representations; in fact, apart from Tina's references to the graphical display those representational tools did not figure into this episode at all. Instead, these segments illustrate students' engagement with multiple displays and inscriptions as resources for establishing shared meaning of the mapping between plain and ciphertext characters. In Tina and Jessica's case, this shared meaning involved establishing common interpretations across different screen views and representational tools. In Tina and Monique's case, establishing shared interpretations of the ciphertext relied on simultaneous examination of common displays—and even a common device. All of these efforts to negotiate shared meaning for the Code Breaker tools were attempts to coordinate multiple participants and displays into effective arrangements for undertaking collaborative decryption tasks. The next episode illustrates the ways that these students would come to enact those alignments in subsequent tasks, and to extend them to inferences about candidate function parameters.

Episode Four

The following excerpt rejoins the group in their next decoding session two days later, and finds Monique and Tina putting their newly-calibrated interpretations of an encoded message together to draw inferences similar to those in the previous episode about the mapping between a one-letter ciphertext value and a plaintext letter. It also finds the group moving into a new arrangement of representational tools and participants as a result of their interpretive work in the previous two episodes. That new configuration would have significant consequences for their efforts to connect distributed representations of the ciphertext with the specification of candidate function parameters. On this day, Tina again had responsibility for the Equation and Graph and Monique the Word Frequency Table as they examined a message encrypted with the function $y=x^2+7$. This segment finds these two students engaged in efforts to establish a shared interpretation of the information provided by the latter representation:

78. Tina: [looking at Graph] All right. Obviously, due to this, it might be a squared but I'm not sure…No, it's not a square. It's not a square… [points to Monique] OK, word frequency. Any one letter-words?
79. Monique: [leans forward and looks closely at the WFT] OK, there's two.
80. Tina: Two one-letter words?
81. Shirley [Taps on Monique's desk] Hey!
82. Monique: [To Tina] Um-hm… It's 8 and… It's 8 and 88.
83. Tina: 8, 88…
84. Monique: And then it's…where it says count, it says two for both of them.
85. Tina: 88 is obviously I.
86. Monique: [holding her stylus up, waves her hand twice from left to right over her PDA as she speaks] Yeah, because it's bigger.
87. Tina: And 8 is obviously…
88. Monique: It's A.
89. Tina: A.

Summary. This excerpt illustrates changes in both individual and collective reasoning about a representational tool. The two distinct one-letter words that Monique identified in the Word Frequency Table (line 82) clearly represented distinct plaintext letters, and the students knew from prior discussions that two commonly occurring single-letter words in the English language were “A” and “I”. Their efforts to articulate which of these plaintext letters mapped to 8 and which to 88 illustrate how much more closely aligned their interpretations of these values had become since Episode Two. As Tina rightly asserted that 88 must be “I”, Monique immediately followed by providing the explanation for this insight (line 86), even using a gesture reminiscent of Tina’s in line 68 of the previous episode to illustrate the relative values. Similarly, as Tina began, “eight is obviously…” (line 87), Monique finished her sentence for her (line 88). Thus, from an individual standpoint, Monique had clearly come to interpret the mapping between plain and ciphertext characters in a way that matched Tina’s. And from a collective standpoint, the students had now established sufficient shared meaning for that mapping that they could now coordinate those interpretations while viewing different displays. Indeed, in the previous two episodes, Tina repeatedly compared information reported from displays examined by other students with that provided in her own view of the Coded Text. By contrast, in this segment she relied exclusively on Monique’s view of the Word Frequency Table, allowing the group to adopt a more efficient distribution of participants and representational tools.

That increasing efficiency at the group level, however, would have important implications for the collaborative participation of all students in subsequent stages of the problem-solving process. Over the next several minutes, Tina worked alone, with paper and pencil, while the other students in the group continued examining the code with their handhelds. The next excerpt opens with Tina explaining her efforts to the rest of the group:

90. Tina: I'm trying to find all possible… I'm writing down all possible functions for A to equal eight.

But…
91. Monique: Wow.
92. Tina: I has to equal eighty-eight at the same time. So I'm crossing out all of these functions that I've thought of so far.
93. Jessica: (nods)
94. Tina: So why don't you guys try and guess and check while I'm doing this.
95. Monique: I thought she said…
96. Shirley: I'm trying to make something.
97. Jessica: You could have told me before.
98. Monique: I'm trying to make something too.
99. Tina: All right. Um…oh no, that wouldn't work.
100. Monique: I'm trying to…
101. Jessica: I'm going to change the function.
102. Shirley: No, I'm going to do it!
103. Monique: I'm going to tap on the line.
104. Tina: This would work. This would work.
105. Monique: Oh, no I'm not going to tap it.
106. Tina: Try y equals ten x minus two.

Summary. As her account suggests, Tina had been writing down a list of linear candidate functions, and checking to see which would include both the ordered pairs (1, 8) and (9, 88), essentially attempting to solve the system of equations 8=a(1)+b and 88=a(9)+b by trial and error. First fixing a value for a, she would generate a list of equations with that coefficient and varying values for b—y=ax+1, y=ax+2, and so on—and then systematically eliminate all those functions that failed to generate both ordered pairs. Eventually, she found a function, y=10x–2, that solved the system of linear equations she had established, successfully mapping one to eight and nine to eighty-eight.[9] In doing so, she abandoned the representations in the *Code Breaker* software altogether, instead relying on her own generation of a series of paper-based inscriptions to manipulate candidate function parameters algebraically. Moreover, she simultaneously abandoned reliance on her group mates. Even as Tina explained her approach, and suggested that they might "try guess and check" while she continued, Jessica expressed frustration (line 97) that Tina had not shared this line of inquiry earlier, and announced (line 101) that she would take

[9] In fact, this function did not solve the code, because Tina had incorrectly assumed that it should be linear; after a hint to this effect from the teacher she quickly came up with the correct quadratic encoding function.

charge of the function-editing responsibility that Tina had abandoned. Shirley and Monique, each in tones suggesting playful sarcasm, asserted that they were engaged in important projects of their own (lines 96, 98, 100-103).

Analysis. Figure 11 illustrates the new arrangement of representational tools and students emerging in this episode. In contrast to prior episodes, the group had now begun drawing inferences about the encoding function parameters not only on the basis of information embedded in features of dynamically linked candidate function representations such as the graph, but also from static problem representations. Moreover, these inferences from ciphertext to candidate Equation parameters were mediated by Tina's use of inscriptions to generate her own paper-based representations of the candidate function. This new alignment quickly became stable; Tina used a combination of paper-based, spoken and gestural forms of symbol manipulation and algebraic reasoning along the lines displayed here in 6 of the group's final 9 decryption tasks. She departed from these approaches only in cases where the code was simple enough to be solved without them, or where a one-letter word did not occur in the message. In each case, she engaged these algebraic representations alone, though the other students continued to provide input from other representations as Monique did here. These algebraic approaches proved both effective, in the sense that they broke several more difficult codes, and efficient, in that in contrast to prior configurations, they could be enacted by a single student.

[Insert Figure 11 about here]

This collapse from a distributed problem-solving system of multiple participants and coordinated representations to the efforts of a single student to complete critical portions of these later tasks represents a clear breakdown in the intended collaborative design. It bears noting, however, that this pattern of frequent solo work by Tina was not established until the 8th of 15 such tasks completed by the group, and that Tina was a year older and two years more advanced in prior mathematics course-taking. In other words, the episode and other subsequent work by this group indicate clear limits in the extent to which this design supported ongoing participation from multiple students, but it does not reveal the relative degree to which these limitations relate to the representational tools and tasks, the ways successive tasks were sequenced, or the level of variation in student knowledge and expertise accommodated by the design.

Summary of Episodes One to Four

Over the course of these four episodes, Group One moved towards more accurate and more efficient ways to configure itself in seeking information about the encrypted message and drawing inferences about function parameters. To catalyze these successive reconfigurations, the students overcame some initial difficulties in interpreting and coordinating their use of the *Code Breaker* representational tools, gradually constructing ways to making shared sense and effective use of the relations among multiple candidate and problem representations. Episode One shows students using the dynamic links between candidate function representations without establishing corresponding interpretive links between the phenomena represented by each display—they made *use* of the fact that the Graph and Inverse Function Table were linked to the Equation in order to complete the task, but there was little evidence to suggest that they made *sense* of those linked displays as multiple representations of a common function. Episode Two likewise highlights how students examined a variety of ciphertext representations but struggled to establish common interpretations of the same set of numerical values across different participants and different displays. Over the course of Episodes Two, Three and Four, members of Group One gradually came to agree on ways to identify, interpret and draw inferences about encoding function parameters from distinctive characters in the ciphertext, and to coordinate those interpretations and inferences across different and distributed representational views. They "learned to see" pattern in and through the representational tools (Goodwin, 1994; Hanson, 1958; Stevens & Hall, 1998). In several instances, the group-level reconfiguring of participants into more effective arrangements took the form of peer-teaching moments, as when Jessica and Shirley in Episode Two and Tina in Episode Three took care in establishing shared interpretations with Monique. In Episode Four, however, moving toward increasing accuracy and efficiency on the part of the group corresponded to participation from fewer individual students.

Episode Five

The following excerpt joins Group Two ten minutes into their efforts to break a code encrypted with the function y=5x+7. During those ten minutes, the group developed a clever way to interpret the rounding bug in the inverse function table as they evaluated candidate functions. Despite the novelty of this discovery, they had been unable to translate it into a successful approach to determining parameters of the encoding function. Jason was responsible for editing the candidate function and observing the Graph in this session, CJ for examining the Inverse Function Table, and Vince the Frequency Tables. Just prior to the start of the excerpt, and without consulting or explaining his reasoning to his group mates, Jason had adjusted their candidate function to y=17x-29. This segment finds CJ reacting to the appearance of that candidate in the Inverse Function Table, and then directing his peers to join him in seeking additional insights from other representational tools. Our analytic focus in examining this episode will be on whether the group established and maintained a stable configuration of participants and tools, and on the ways they interpreted relations among different representations.

107. CJ: [Looking at the IFT as Jason is editing candidate function] OK, you're definitely going far off. It's AA, CC, DD, E, FFF, G, HHH, J. [Watches IFT for 10 more seconds before scrolling up to look at the Graph] OK, can I show you something?
108. Vince: What?
109. CJ: You guys scroll up to the…graph. OK?

110. Vince: [scrolling up] OK.
111. CJ: [taps the "H" label on the x-axis of the Graph to highlight trace line] Click on H.
112. Vince: [taps the "H" label on his own PDA] H.
113. CJ: See how they're, like, the lines hit it and branched off? That's bad. We want to get rid of that. [to Jason] You need to make that negative number there, your -29 higher. And your 17 lower.
114. Vince: Hey, change that constant.
115. CJ: Change the constant back to zero. Change the constant back to zero!
116. Jason: I'm trying to! For the negative, do I make it like a higher negative, or a lower negative? More towards a big negative?
117. CJ: More towards zero. Make the 29, start making the 29 more towards zero. And 17 more towards zero as well.
118. Vince: OK, you're getting real close.
119. CJ: You're getting much closer. Hey, Vince. Watch the frequency table, all right?
120. Vince: Frequency?
121. CJ: Yeah. Watch for…every like four or five changes,
122. Vince: [Scrolls to Frequency Table] Uh-huh.
123. CJ: Or when I ask you, tell me the most common numbers. The most common Y, OK? When I ask you to, tell me the most common Y.
124. Vince: Right now, it's 112 and 22.
125. CJ: And how many are there?
126. Vince: Four.
127. CJ: Four.
128. Vince: [Scrolls to WFT] There's two words that are the same.
129. CJ: OK, hold on, hold on. Uh, Jason, hold on.
130. Vince: Wait a sec. Hey, there's two words that are the same, they both start with 12, and 12 is one of the most frequent numbers.
131. CJ: We got it. We're on the right track. Look at the inverse frequency [sic] table.
132. Jason: [tapping the "L" label on the x-axis of the Graph] Wait. L right now is 12. And that's the most common.
133. CJ: Everybody. Go to inverse frequency [sic] table. [Vince and Jason both scroll to the IFT] We have a B, C, E, F, H, I, J, K, L, M, N, O, P, Q, R, anyway. That's good because there are more ones.

Summary. This episode finds the group moving among several representations of both the encrypted message and the candidate function, and coordinating the attention of several participants toward each of those displays. They examined five different representational tools here in a span of less than two minutes, and in several instances individual students adjusted their viewing windows in order to jointly consider and seek to establish shared interpretations of different displays. CJ's inferences from the Inverse Function Table (lines 107, 131, 133) illustrate the group's opportunistic use of an accidental feature provided by the buggy behavior of this tool, rejecting candidate functions that appeared to imply the mapping of a single input to multiple outputs.[10] The students then examined the graph, and drew the same conclusion from a graphical manifestation (Figure 12) of the same buggy phenomenon (lines 107-113). Vince then scrolled to and made observations based on the Character (lines 124, 126) and Word (lines 128, 130) Frequency Tables, while Jason followed up on Vince's report from the frequency tables by tracing the mapping of L to 12 through the current candidate function (now y=x) on the Graph (line 132).

[Insert Figure 12 about here]

Analysis. Figure 13 shows the arrangement of students and representational tools adopted by Group Two over the course of this excerpt. As with Group One's early decryption efforts in Episode Two, that configuration was complex, featuring three students[11] jointly engaging several representations in succession. The group devoted considerable effort to developing and coordinating ways of interpreting representations of both the candidate function and the ciphertext, but they were not yet successful in establishing interpretive links between those different sets of representational tools. They drew inferences about parameters of the algebraic candidate function from both the inverse function table and graph, though neither of those efforts resulted in the correct determination of an unknown parameter, and both relied on the buggy behavior of the software rather than any characteristics of the ciphertext values. And they also examined and discussed two ciphertext displays, though they were not able to relate those observations to the encoding function. Indeed, rather than a stable alignment of participants and tools, the group's strategic work in this segment was characterized by shifting views and exploratory investigations of different representational tools.

[Insert Figure 13 about here]

[10] See White (2008) for a detailed analysis of this approach.
[11] The group's fourth member, Reggie, was absent on this day.

Notably, however, the students' efforts to make sense of these different representational tools in relation to the decryption task did establish a number of temporary interpretive links between displays. Immediately after observing in that the letter H, among others, was associated with three different values in the ciphertext, CJ shifted his view to the graph, and used a feature of that display to examine the same letter from another view (line 111). As a consequence, he witnessed the same non-functional behavior (associating a single input with multiple outputs), and interpreted it in the same way, as something "bad" that they needed "to get rid of" (line 113). A moment later, he sought to establish and explore the potential utility of another link between representations, asking Vince to observe the frequency table and "watch for" changes in common characters corresponding to "every like four or five changes" to the candidate function parameters (lines 121, 123). In this case, and reflecting the students' still-limited familiarity with the behavior of and still-developing meanings for these tools, he expected a dynamic relationship between displays when there was none. And Jason sought to extend Vince's analysis of the ciphertext character 12 by locating it in the graph, though he appeared uncertain about how to interpret the mapping between that character and the letter L indicated by the current candidate function (line 132). In each case, the students appear to have been trying to make sense not only of the intended dynamic links among multiple representations of the candidate function, but of a much wider array of possible relations among representational tools, displaying varying degrees of success and considerable stores of both confusion and creativity as they went. The next two episodes illustrate the ways they would come to establish and work to sustain ways for interpreting and applying one such set of relations among displays.

Episode Six

Over the final few class sessions of the study, the students in Group Two developed a configuration of representations and participants that they would come to use with considerable effectiveness in breaking some challenging codes. In the next episode, they faced a message encrypted with the function $y=12x^3-15$. During this session, Jason edited candidate function parameters, while Reggie was assigned to watch the graph, Vince the Function Table, and CJ the Frequency Tables. Our analysis again focuses on whether and how the group stabilized alignments and interpretations of representational tools, this time with particular attention to the ways their approach involved coordinated efforts among multiple participants. As Jason adjusted the candidate function parameters, Vince examined the ciphertext values displayed in the frequency table, and then began comparing with those in the function table:

134. Jason: So higher, or lower, what? [Changes exponent from 3 to 2, then back to 3 again] It's probably three. [begins increasing the coefficient]
135. Vince: [Looking at Function and Frequency Tables as coefficient changes to 12] OK. Stop, right there. You're within range.
136. Jason: All right, so keep on doing the y? [Begins increasing coefficient again]
137. Vince: So, change the, change the constant a bit. Wait, let me see something.
138. Jason: OK, yeah, the constant.
139. Vince: The smallest number… wait, stop. Let me think. Let's see...If…if, if, if, if, if, if…[leans over his PDA, alternately moving his stylus toward values in the function and frequency tables for approximately 12 seconds, then looks up] What's the, what's the coefficient at right now?
140. Jason: The coefficient, um, it's at 13.
141. Vince: 13?
142. Jason: Uh-huh.
143. Vince: Change it to 12. [Waits for Jason to edit value, and then reacts when his view updates] OK.
144. Jason: Wait. It's at 11. There we go.
145. Vince: OK, 12. Now change…what's the...constant at?
146. Jason: (glances at poster) Constant's at 7.
147. Vince: 7? Change it to…let's see. Change it to 6.
148. Jason: Alrighty.
149. Vince: 5…

Summary. This excerpt finds both Jason and Vince working to draw inferences about candidate function parameters from their respective views of the Graph and the Function and Frequency Tables. From his view of the Graph (Figure 14a), Jason inferred a cubic function with a large coefficient (line 134). As Jason increased the coefficient, Vince watched the changing values in the function table and compared them to those in the frequency table (Figure 14b), stopping Jason when the values came "within range" of one another (as, for example, with the values on the order of 187,000 and 210,000 that appeared in the function table to match those of the same order just below in the frequency table as Jason adjusted the coefficient to 12), and directing him to next lower the constant to bring those values still closer together (lines 135, 137). In fact, changes on one student's device took a few seconds to update on those of the others in the group, so in this instance Jason had already changed the coefficient to 13 before Vince told him to stop, and clarifying the actual value required some coordination (lines 139-145). Having resolved the coefficient, Vince directed Jason to lower the constant from 7 to 6, and then to 5, each time watching the corresponding transformation of all the y-values in the Function Table one unit closer to those in the Frequency Table as they would gradually converge on the solution.

[Insert Figures 14a and b about here]

Analysis. This emerging approach to using the tables in combination would prove to be an effective new organization of participants and representational tools. Vince and Jason managed to draw inferences about candidate parameters from this alignment that were highly accurate, and that would subsequently allow them to successfully complete the task. The utility of this approach can be understood in terms of the way it matched representations of the candidate function and the encrypted text, as illustrated in Figure 15. This distributed problem-solving approach closely resembles the configuration enacted by Group One in Episode Four (compare Figures 11 and 15). Just as Tina matched ciphertext values displayed in the Word Frequency Table with plaintext inputs through an algebraic representation of a candidate function, here Vince and Jason matched ciphertext values in the Frequency Table with plaintext values through a tabular representation of the candidate function.

[Insert Figure 15 about here]

However, while Tina did that work in isolation, Group Two's approach required two students simultaneously viewing devices positioned to different *Code Breaker* displays. There was a notable division of labor organizing the approach; while Jason examined the graph and inferred values for the exponent and coefficient, Vince did the work of interpreting the relationship between the tables and drawing inferences about the constant. On the next day of class, the role assignments for these two students would be switched, so that Vince edited the candidate function while Jason viewed the frequency and function tables. As the next episode illustrates, in order to maintain the same problem-solving configuration despite a different distribution of representational tools among participants the students would first have to carefully coordinate their interpretations of those various displays.

Episode Seven

In the segment below, students' assigned representational views were swapped from the previous episode so that Vince now edited the Equation while Jason watched the Function Table. In this final excerpt, the two students worked to coordinate their interpretations of these representational tools so that they could enact the same strategy as before, but in these reversed roles. Here, our analysis focuses particularly on how interpretations of the links between multiple and distributed representations are established and maintained through interactions between multiple student participants. In this task, the message had been encrypted with the function $y=19x^3+1$. Prior to the following excerpt, the group had already used the graph and ciphertext to deduce a cubic candidate function with a lead coefficient in the high teens. Upon editing the lead coefficient to a value of 18 based on his view of the graph, Vince asked Jason whether the values in the resulting function table were approaching those of the frequency table:

150. Vince: Am I close?

151. Jason: [looking at the Frequency Table] You have a frequency for…nothing.
152. Vince: What the…can I see this? (Reaches for Jason's PDA) OK, I…duh. I am way off. [Points to the Frequency Table on Jason's PDA] OK, what's the highest number here?
153. Jason: One…
154. Vince: [Looks at the Function Table on Jason's PDA] I'm too high.
155. Jason: [Looking at the Frequency Table] Huge.
156. Vince: Too high.
157. Jason: [Comparing Frequency and Function Tables] All right, you're pretty close.
158. Vince: I am?
159. Jason: Go back.
160. Vince: Here's what I want you to tell me. OK. [Pointing to Frequency Table on James's PDA]. Right here, OK, 39556, right? Look for a number close around.
161. Jason: Like, right here [points to Frequency Table], before when it was three hundred thirty-three thousand something it was pretty close to this.
162. Vince: OK. Tell me if I'm close. How's that?
163. Jason: Uh, mine hasn't loaded yet.
164. Reggie: Pretty close.
165. Jason: Yeah, it's close.
166. Vince: Do I have any uhh…
167. Jason: Yeah, you have a match, right here [points to Frequency Table].
168. Vince: I do? One?
169. Jason: Yeah.
170. Vince: OK. How about now?
171. Jason: Wait. Um, it's a little bit off… Keep on going… Off, down. We're pretty close.
172. Vince: [Scrolls down to compare Frequency and Function Tables himself] OK. Our lowest number is 24, then the real lowest number is…one hundred…eh, no, the real lowest number is 20. I'm going… OK, we got it.

Summary. Vince's opening question (line 150) referred to the closeness of the fit between the values in the frequency table, and those in the function table based on the candidate he had just edited. Jason's reply (line 151), however, was an attempt to interpret other information presented in the latter representation. Each student had a hand on Jason's computer as they sought to coordinate their use of these representations by pointing to a shared display (lines 152-156). As Vince returned to his own device and adjusted the candidate function according to this latest analysis, Jason reported in line 157 that they were now "pretty close". Vince's questioning reply (line 158) reflects his surprise at this comment, and even as Jason gave a corresponding instruction (line 159) for editing the candidate, Vince proceeded to conduct another instructional sequence (lines 160-2) to make sure that they were interpreting these representations in the same way.

At this point, the students had sufficiently calibrated their manipulations and interpretations of two devices and three representations—the candidate function and the frequency and function tables—that they were ready to apply this distributed array to finishing the decryption. Jason reported that he had found a pairing of identical values in the function and frequency tables, and then continued to provide feedback as Vince adjusted the coefficient to 19 and sought an appropriate constant value (lines 167-171). Comparing the lowest values in the function table (Figure 16a) and frequency table (Figure 16b), Vince then reduced the constant from five to one to bring the values into alignment (line 172).

[Insert Figures 16a-b about here]

Analysis. The arrangement of representations and participants in this episode was as in Figure 15, but with Jason's and Vince's views switched (and with Vince sometimes doubling up with Jason's views of the tables in order to coordinate interpretations). This episode illustrates the ways these students established an interpretation of the relationship between the frequency and function tables, and thus of problem and solution representations. It also highlights the ways distributed representations of the ciphertext served as resources students used to interpret the links among multiple function representations. In other words, Vince and Jason found ways of making the Equation and Function Table meaningful in relation to one another precisely because they devised a way of manipulating and interpreting these multiple representations of a common function in simultaneous relation to a representation of the encrypted text.

Vince and Jason's approach to combining these representational tools required two students to operate simultaneously, as it involved one student viewing and editing the candidate function while the other compared the changes in the function table with the static frequency table display. In this episode, Vince actually performed both these roles himself in three instances—

twice by looking at Jason's device (lines 152-156, 160) and once by changing the view on his own screen (line 172). But in each case his doing so reflected temporary inefficiencies that would allow the two-student configuration to operate effectively. In particular, most of Vince and Jason's conversation in this episode was directed toward establishing a shared interpretation of the function and frequency tables in relation to the decryption task. Doing so hinged on their convergence toward collective meanings for the word "close" and the notion of a "match" as each provided metrics for aligning the function and frequency tables, and thus for aligning the known candidate and unknown encoding functions. Rather than attempts to bypass collaborative interactions, Vince's reaches toward Jason's PDA both appear to have been efforts to establish this shared meaning and the distributed problem-solving efficiency it would ultimately support. In each case, Vince accompanied those reaches by indicating respective places in each table where Jason might compare values. And the instance in which Vince scrolled down to view the tables on his own device came in response to Jason's identification of a "match" (line 167) and his description of the candidate value as "close" but "a little bit off" (line 171). In each case, the students sought to converge on a shared set of discursive tools that would allow them to efficiently and effectively coordinate their joint use of three representations.

Summary of Episodes Five to Seven

As with Group One, Group Two gradually developed ways of configuring participants and tools in order to establish ways of interpreting multiple representations of the candidate function in relation to distributed representations of the encrypted text. Episode Five illustrates how students actively sought links between representational tools even in initial code breaking trials, but these early efforts to identify relationships between displays did not necessarily correspond to the actual dynamic links between representational tools established by the software, or prove useful in relation to the decryption task. In particular, the group was not yet able to form the kind of interpretive link between a representation of the candidate function and a representation of the encrypted text necessary for reliably solving codes. Over the course of Episodes Six and Seven, however, students in Group Two moved to a more stable and more reliable alignment of multiple participants and tool, devising an effective approach for coordinating interpretation and use of the Frequency and Function Tables and the Equation across two different student views. Moreover, unlike Tina's largely independent efforts to link representations of the candidate function and encrypted text by the end of Episode Four, Group Two's approach to comparing such representations featured ongoing coordination between multiple students.

DISCUSSION AND DESIGN CONSIDERATIONS

Over the course of the unit, students in the two focal groups reorganized themselves into more efficient and more effective arrangements of participants and representational tools. As they worked to adapt to the challenges required to complete increasingly difficult tasks, both groups moved from relatively uncoordinated and inconsistent uses of multiple displays toward the establishment of more stable and successful ways of coordinating participants and aligning representations of the encrypted text with those of a candidate solution function. These changing configurations required students to interpret representational tools in relation to the decryption task, and to collectively coordinate those interpretations in order to effectively combine representational tools across multiple participants and devices. Below, we consider these successive reorganizations first in terms of the ways they hinged on students' efforts to reason with representational tools in relation to the problem-solving task, and then in terms of the extent to which they afforded collaborative interactions or necessitated coordination among multiple participants.

Reasoning with Representational Tools

These reorganizations at the level of the distributed problem-solving group were emergent from students' individual and collective efforts to reason with representational tools. From the outset, students actively sought to establish interpretive connections both among multiple linked function representations, and between function and problem representations. However, groups' collective interpretations and effective alignments of these distributed representational tools were built up slowly, and required considerable coordination among multiple students over a series of decryption tasks.

Students' interpretations of these representational tools reflected their relevance for particular approaches to solving the decryption task, rather than their meaning in relation to the cryptographic context or the underlying function. So, for example, they often used the Word Frequency Table not to draw inferences about the characters making up commonly occurring words in an encrypted message (designer's intent), but rather to identify single-letter words in the cipher text. And they used the Character Frequency Table not only to analyze and seek to identify commonly occurring letters (designer's intent), but also to identify high and low values in the range, and to compare and match cipher text values with those generated by a given candidate function. And they used the graphing tool not only to generate the curve associated with a particular candidate function within a fixed display window, but also as a way to further examine phenomena that they had observed in other displays, and to demonstrate those phenomena to one another. Thus, in Episode Three, after Jessica reported a value as a single-letter word in the Word Frequency Table, Tina followed up by locating the same value on the y-axis of the Graph and using its relative location to justify her conclusion that it was mapped from an I. Later in the same episode, Tina likewise used the distribution of plaintext letters along the x-axis as a resource for explaining her interpretation of the mapping between those letters and the ciphertext characters to Monique. And in Episode Five, CJ and Vince used the Graph to trace the mapping of a single plaintext letter to multiple ciphertext characters in order to confirm a phenomenon they had observed in the Inverse Function Table. Later in that episode, Jason likewise sought to follow up on Vince's report of a commonly occurring character in the frequency tables by tracing it through the graph of the candidate function.

These varied uses suggest that while students did not always derive a particular intended meaning from a given representation, they did demonstrate considerable competencies for interpreting, applying and making connections across those representations as problem solving tools. And in fact, in cases where students' attempts to use these representations in their intended ways—i.e. as frequency displays—were not aligned with the strategic configuration of the group, they were either ignored (as in Jessica's report in line 38 that "Negative forty…happens two times in the poem" and Monique's similar observation in line 84 about the "count" for the two one-letter words she found), or taken by other group members as occasions for recalibrating collective interpretation (as in Jason's report of a "frequency for…nothing" in line 151). Indeed, it was through the establishment of emergent, task-specific—rather than canonical or designer-intended—meanings for representational tools that groups ultimately reconfigured participants and representations into increasingly effective problem-solving ensembles. In other words, harking back to our theoretical framing at the outset of this paper, it was students' efforts to interpret these displays *in relation to a problem-solving task,* rather than their apprehension of any standard meaning for a canonical representational form, that led to their reorganizations into new arrangements of multiple representational tools for solving that task.

Although students did routinely capitalize on the dynamic links between multiple function representations—particularly those between symbolic and graphical displays—as resources for breaking codes, it often appeared that they made use of those linked displays without thinking of them as corresponding to different representations of a common function. This finding is consistent with earlier research and theory highlighting the unobvious linkages between different representations of mathematical functions even if dynamically linked in computer displays (Kaput, 1993; Schoenfeld et al., 1993; Thompson & Sfard, 1994). Indeed, there was little reason to expect students to interpret the links between function displays in this way; they were an automatically propagating feature of the software environment, such that the connections among multiple representations were made by the device network rather than needing to be made by the students—these links were 'offloaded' in this distributed cognitive system (Pea, 1993). Rather, students *capitalized on* those connections in order to align representations of the problem and a solution and thus complete problem-solving tasks. It was these links between distributed problem and candidate solution representations—between the left and right circles depicted in Figure 2—that required students to engage in efforts to reason about the mathematical meaning and the potential utility of different displays in relation to the problem-solving task, and to engage in collaborative interactions with one another through which they coordinated interpretations in order to enact decryption strategies. In this sense, multiple linked representations served as resources not for illuminating or building understanding of the common underlying referent they shared—the mathematical function—but rather for coordinating interpretations of different mathematical displays across multiple participants and devices.

To summarize these findings: student reasoning with the multiple and distributed representations provided in the *Code Breaker* environment did not take the form of discovering or enacting either the designer-intended meaning of each representational tool or the full set of designed relationships among those tools displayed in Figure 2 in the process of completing any particular problem-solving task. Instead, learning for these students occurred through the repetitive encounter with an increasingly difficult sequence of mathematical tasks with the same representations, which provoked successive reorganizations at the level of the group between tasks.

This notion of distributed learning—as adaptive reconfigurations of participants and tools—may mark an important contrast between the present design and that of Hutchins' renowned exemplar of distributed cognition provided by large ship navigation. In "the wild," distributions of intelligence are emergent properties of complex social and material systems that have developed over long cultural histories (Lemke, 2000)—centuries, in the case of ship navigation. In contrast, the present environment begins with an intentional rather than an emergent arrangement of participants, tools and task—a system *distributed by design*, much like the engineering of a classroom "community of learners" in the work of Brown, Campione and colleagues (Brown et al., 1993; Brown & Campione, 1994). And whereas successfully steering a large ship may require patterned enactment of particular roles and responsibilities, successful problem-solving performances and individual student learning opportunities in the *Code Breaker* case appear to depend more on the successive reorganization of elements in that system by participants. Indeed, as Bazerman (1996) notes, in the Hutchins example "[t]he cognitive task is to align the self with narrowly defined pre-determined roles and behaviors—a very different situation than most studies of collaborative work, in education or industry, where the emphasis is more on individualized and improvisatory behaviors to meet personal perceptions and needs" (p. 53). Thus, while much of human activity on broader timescales involves reifying effective forms of practice and stabilizing efficient divisions of labor, the relevance of distributed cognition to classroom collaboration may hinge on the extent to which learners have ongoing opportunities to adaptively reorganize ensembles of human and material resources in the dynamical system in which they participate[12].

Distributed Problems and Collaborative Tasks

Importantly, these different configurations at the level of the distributed group provided correspondingly different opportunities for individual students to reason with representational tools and participate in collaborative learning processes. In some cases these reorganizations found the groups moving toward problem-solving efficiencies that minimized the necessary number of representational resources, and often also the number of participating students. For example, over the course of Episodes Two through Four, Group One moved from four students seeking and trying to interpret the same information across several displays, to two students seeking and interpreting similar information from a similar display, to one student drawing inferences from that information independently. On the other hand, the process of reorganizing the group into more effective arrangements was a central source of learning opportunities for individual participants, particularly when, as in the cases of Monique in Episodes Two and Three and Jason in Episode Seven, those arrangements relied on coordination of meaning for a given representation in order to allow consistent interpretations when students did not share a common view. In each of these cases, the group's efforts to bring participants' interpretations into alignment across different views marked learning opportunities for individual students.

[12] In this sense, the relevant analog from Hutchins' analysis to classroom learning activity is not the routine of the fix cycle, but rather the crisis prompted by a failed gyrocompass. Such disruptions of regular practice, so undesirable and potential hazardous in the workplace or in the wild, are precisely the moments to strive for in the design of learning environments.

Ultimately, then, simply having linked multiple representations of a function being focused upon by different members of a collaborating group did not guarantee the collaborative participation of the students in that group, any more than having linked multiple representations of a function made self-evident that they referenced that common function. To the extent that opportunities for student interaction and participation were missed in these instances, they may be partly explained in terms of limitations in the present design. For example, the Equation and Function Table were in different locations in the display window and a significant amount of scrolling was required to toggle between them. On the other hand, the closer proximity of the Equation and Graph allowed them to be simultaneously analyzed by a single participant, even though one student was assigned responsibility for editing the former display and another for observing the latter. This proximity may have minimized the need for students to establish and coordinate collective interpretations across those different displays—in particular, to engage in the kinds of efforts to coordinate interpretations of the Function and Frequency tables and Equation by CJ and Vince, or of values in the Word Frequency Table observed by Jessica or Monique as Tina sought to draw inferences about candidate function parameters.

The empirical consequences for the student groups of these and other design decisions might be interpreted in terms of a tension between affordances for individual users and groups. The proximity of Equation and Graph allowed them to be simultaneously scrutinized—and thus engaged as dynamically linked displays by a single user. By contrast, the Equation and Function Table could only be simultaneously displayed, and so dynamically linked, across multiple students' devices. Similarly, the choice to allow each student to move freely among all representational tools rather than restricting their views to their respective role-assigned displays had important implications for student collaboration. While some students, as illustrated by the participants in Group One throughout Episodes One to Four, tended to adhere closely to their assigned roles and views, others more frequently disregarded those assignments and moved among different views during a task. The latter approach was often demonstrated by Group Two, particularly as several students successively shifted their displays in order to consider common views in Episode Five, and when Vince moved to Jason's view in order to complete the task by himself at the end of Episode Seven. Students did work to coordinate interpretations of representational tools both in instances when they shared common views, and when they examined different displays. So while a more rigid design for differentially distributing views might ensure greater interdependence among participants and yield more opportunities for coordinated conversation across different displays, the more flexible design presented here did appear to have some advantages for allowing students to coordinate interpretations of common displays, and to more fluidly seek out different distributions of participants and displays.

In a similar vein, opportunities for collaboration with multiple representations of functions might be substantially enriched through multiple mechanisms for manipulating those functions. While the symbolic expression of the candidate function—and thus the single student assigned responsibility for that expression—was the 'driver' of transformations across all dynamically linked representations in the present design, and those transformations were limited to discrete and unitary adjustments of each parameter, other multiple representation environments have explored a variety of alternate approaches to user input and control (Kieran & Yerushalmy, 2004). Kaput (1998) stressed the potential of bidirectional links among representations, such that inputs for multiple linked representations allow the direct manipulation of a graph as an alternate means of transforming the expression. Building a networked and collaborative environment with bidirectional links would allow opportunities for each student in the group to simultaneously engage in the generation or manipulation of representational artifacts, and thus might substantially enrich the possibilities for coordinated action and interpretation across different displays.

CONCLUSION

Multiple representations hold great promise for organizing novel learning environments and activities. Multiple representations of function, in particular, have provided the foundation for a variety of alternative secondary mathematics curricula and novel technology-based learning environments that capitalize on the rich conceptual linkages among symbols, graphs, tables and situations. Representational tools and inscriptions also play pivotal roles in mathematical forms of interaction and communication, and in the coordinated efforts of multiple people to accomplish complex tasks using such representations. These intersections of representational affordances and communicative activities provide fertile terrain for fostering participation in mathematical practices, for orchestrating collaborations, and for theorizing about learning.

However, the potentials of multiple representations for advancing learning are not always easily realized. The richness and complexity of representational and collaborative relationships pose challenges for successfully engineering learning experiences. By designing a learning environment for collaboration with multiple representations, and analyzing two teams of students working together in that environment over several weeks, we sought to better understand the conditions for success. Our findings address both theoretical and practical issues in designing for collaboration with multiple representations.

Regarding theory, our analysis foregrounds the need for a perspective on collaborative learning that elaborates on Hutchins' seminal analysis of distributed cognition. Whereas Hutchins' case of cognition 'in the wild' depends on stable roles and subtle arrangements of knowledge artifacts that emerge in the course of the slow development of a cultural practice (ship navigation), our classroom cases argue that students need rapid successions of opportunities to reconsider and reorganize their roles and to experiment with different meanings and uses of flexible tools in the context of tasks with carefully sequenced levels of difficulty.

Moreover, our analysis shows that learners do not always apprehend or appropriate designers' intended meanings either for representations, or for pre-assigned collaborative roles. They do, however, show considerable competence and creativity in finding meanings and uses for multiple representations. Consequently, educators should not assume that collaboration designs with multiple representations will produce intended results, especially in a short-term activity. Such designs should seek a productive balance between the advantages of more fluid and more rigid assignments of unique representations to students, and between individual and group uses of representations. If as in the present case these representations are subtle and complex, it is likely that teams of learners will need an extended sequence of increasingly challenging tasks to both (a) discover effective collaboration patterns and (b) successively reorganize their uses of multiple representations. For both design and instruction, we therefore recommend seeing the links among representations not as fixed by a common referent or following predetermined ways of relating, communicating and coordinating action, but rather as emergent webs of relations to be continually drawn, reorganized, and reestablished through complex collaborative problem-solving activity.

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Figure 1. A student group using the *Code Breaker* software with handheld computers

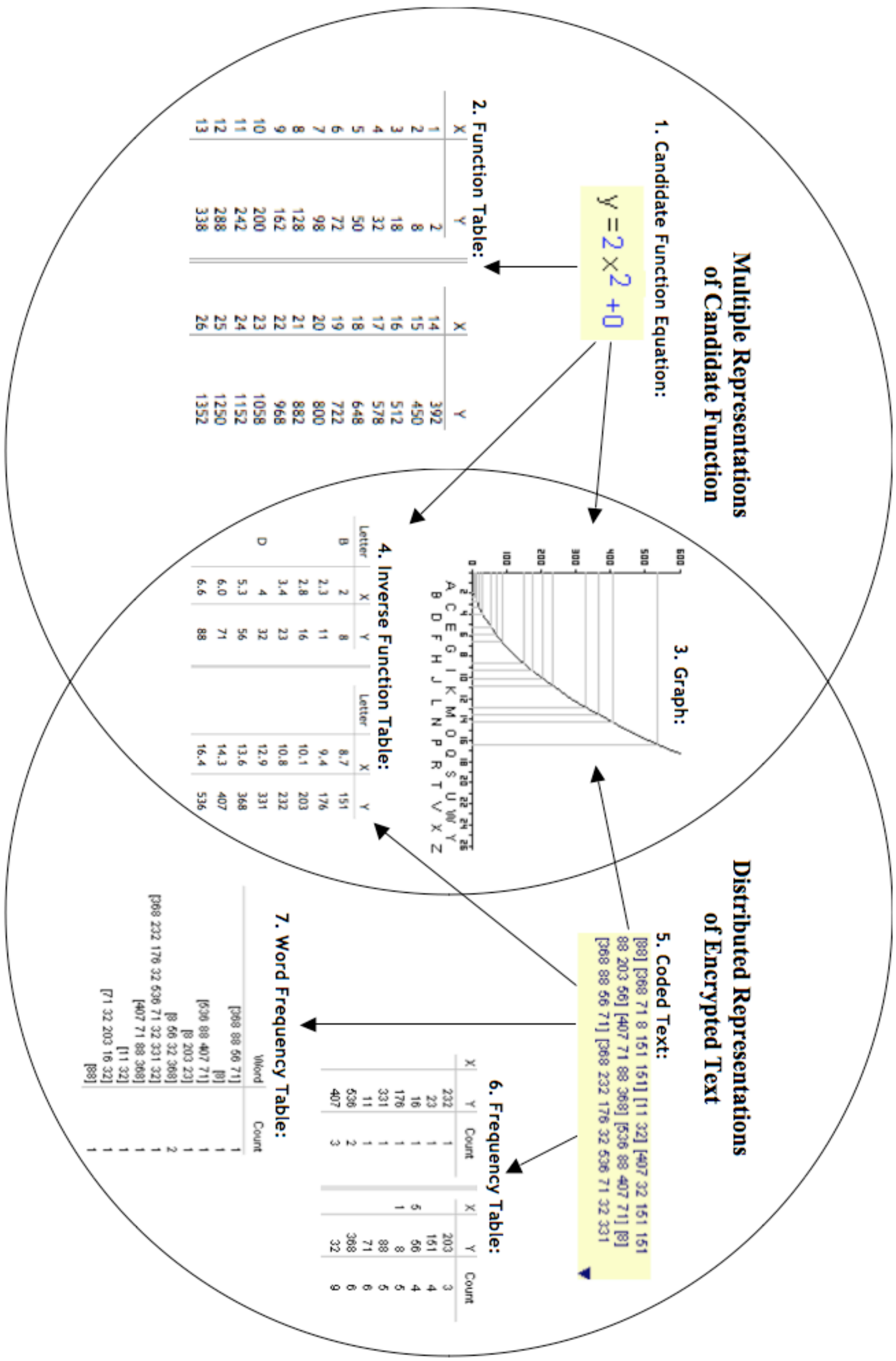

Figure 2. Code Breaker array of representational tools

Letter	X	Y	Letter	X	Y
	2.4	12		5.6	67
	2.7	17		5.9	72
C	3	22		6.5	82
	3.6	32		6.8	87
	4.2	42		7.7	102
	4.5	47	H	8	107
	4.8	52		8.3	112

Figure 3a. Inverse function table

Letter	X	Y	Letter	X	Y
B	2.1	12	E	2.1	67
B	2.1	17	E	2.1	72
C	3	22	F	3	82
C	3.1	32	F	3.1	87
D	4.1	42	G	4.1	102
D	4.1	47	H	4.1	107
D	4.1	52	H	4.1	112

Figure 3b. Inverse function table with bug

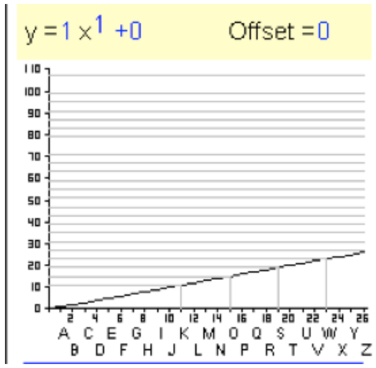

Figure 4a. Graph for y=x

Inverse Function Table:

Letter	X	Y	Letter	X	Y
K	11	11			59
O	15	15			63
S	19	19			67
W	23	23			71
		27			79
		31			83
		35			87
		39			91
		43			95
		47			99
		51			107
		55			

Figure 5a. Inverse Function Table for y=x

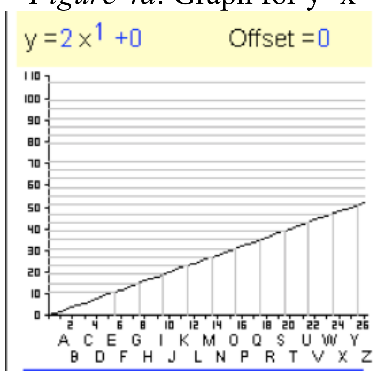

Figure 4b. Graph for y=2x

Inverse Function Table:

Letter	X	Y	Letter	X	Y
E	5.1	11			59
G	7.1	15			63
I	9.1	19			67
K	11.1	23			71
M	13.1	27			79
O	15.1	31			83
Q	17.1	35			87
S	19.1	39			91
U	21.1	43			95
W	23.1	47			99
		51			107
		55			

Figure 5b. Inverse Function Table for y=2x

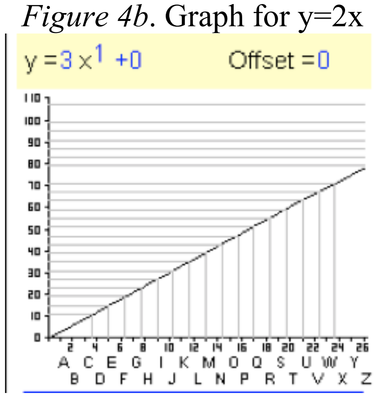

Figure 4c. Graph for y=3x

Inverse Function Table:

Letter	X	Y	Letter	X	Y
C	3.1	11	S	19.1	59
E	5	15	U	21	63
F	6.1	19	V	22.1	67
G	7.1	23	W	23.1	71
I	9	27			79
J	10.1	31			83
K	11.1	35			87
M	13	39			91
N	14.1	43			95
O	15.1	47			99
Q	17	51			107
R	18.1	55			

Figure 5c. Inverse Function Table for y=3x

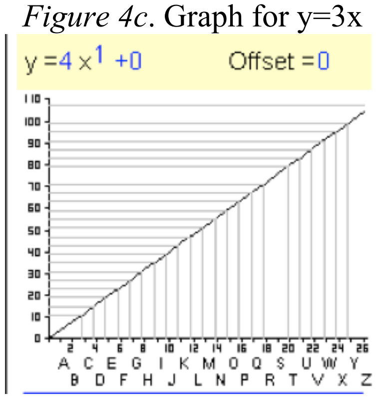

Figure 4d. Graph for y=4x

Inverse Function Table:

Letter	X	Y	Letter	X	Y
B	2.1	11	N	14.1	59
C	3.1	15	O	15.1	63
D	4.1	19	P	16.1	67
E	5.1	23	Q	17.1	71
F	6.1	27	S	19.1	79
G	7.1	31	T	20.1	83
H	8.1	35	U	21.1	87
I	9.1	39	V	22.1	91
J	10.1	43	W	23.1	95
K	11.1	47	X	24.1	99
L	12.1	51	Z	26.1	107
M	13.1	55			

Figure 5d. Inverse Function Table for y=4x

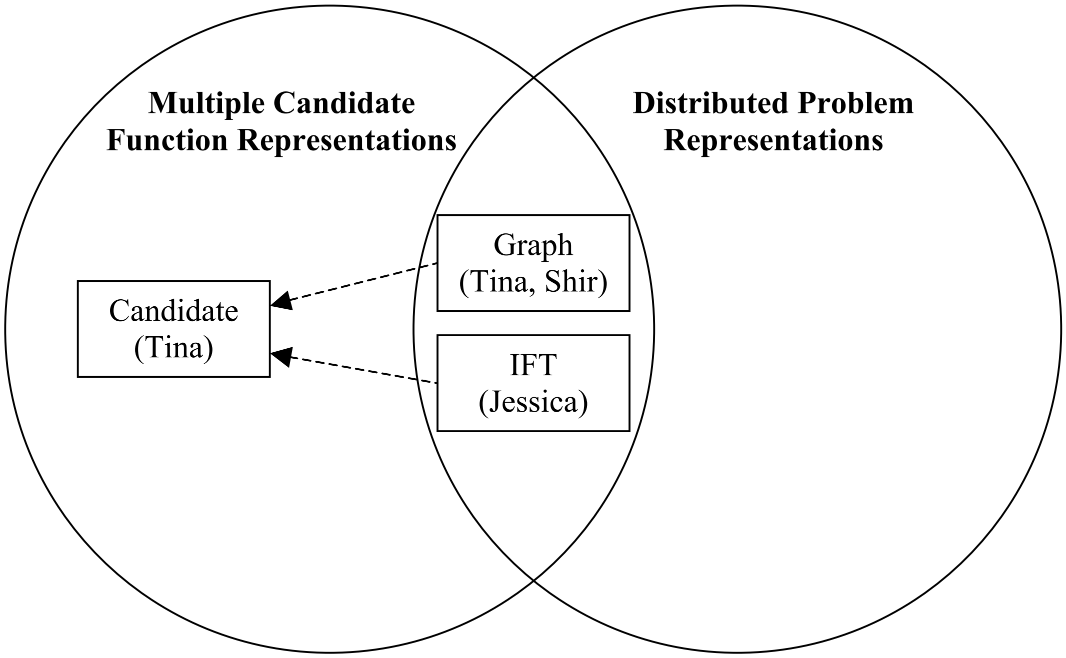

Figure 6. Configuration of Representations and Participants in Episode 1

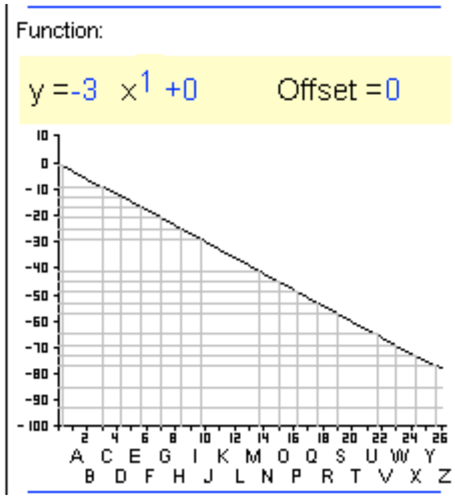

Figure 7. Graph of y=-3x

Word Frequency Table:

Word	Count
[-73 -25 -13 -45]	1
[-73 -25 -53 -77 -21 -25]	1
[-85 -53 -65 -49]	1
[-73 -25 3 -73]	1
[-17 -53 -65]	1
[3 -69]	1
[-73 -25 -13]	2
[-25 3 -9]	1
[-69 3 -45 -13]	1
[-65 -13 3 -41 -41 -93]	1
[-57 3 -69 -69 -29 -49 -21]	1
[-73 -25 -13 -65 -13]	1
[3 -1 -53 -77 -73]	1

Figure 8a. Monique's view of the WFT

Inverse Function Table:

Letter	X	Y	Letter	X	Y
		3			-49
		-1			-53
		-9			-57
		-13			-65
		-17			-69
		-21			-73
		-25			-77
		-29			-85
		-41			-93

Figure 8b. Jessica's view of the IFT

[-73 -25 -53 -77 -21 -25] [3 -69] [-17 -53 -65] [-73
-25 3 -73], [-73 -25 -13] [-57 3 -69 -69 -29 -49
-21] [-73 -25 -13 -65 -13] [-25 3 -9] [-85 -53 -65

Figure 8c. Tina's view of the Coded Text

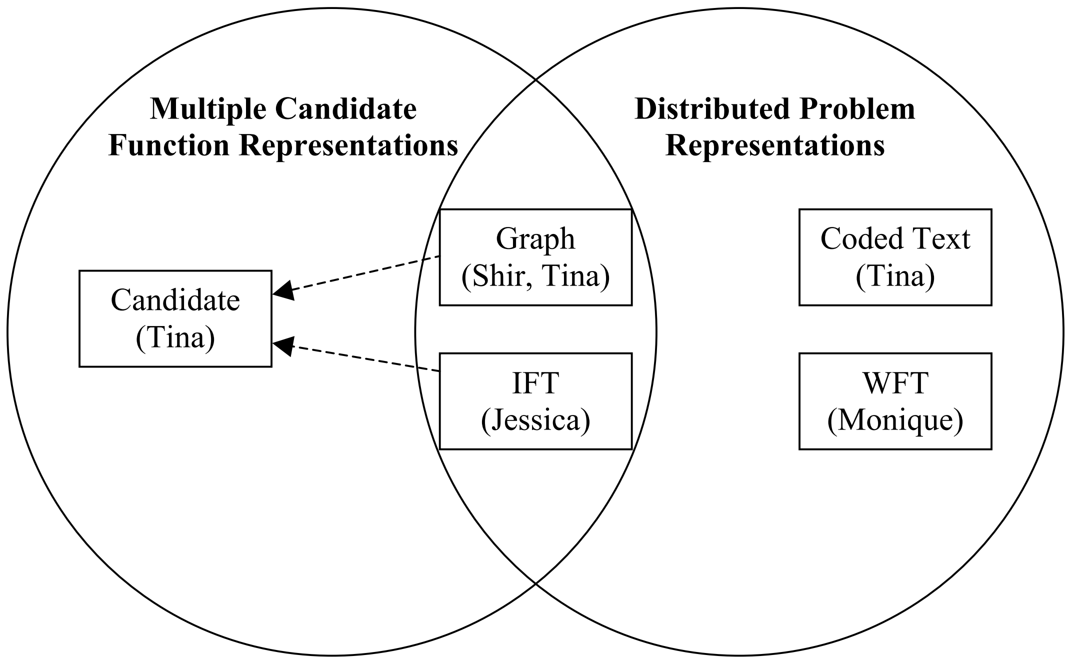

Figure 9. Configuration of Representations and Participants in Episode 2

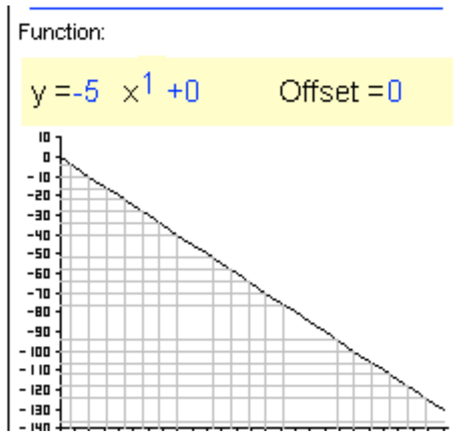

Figure 10. Tina's view of the Graph

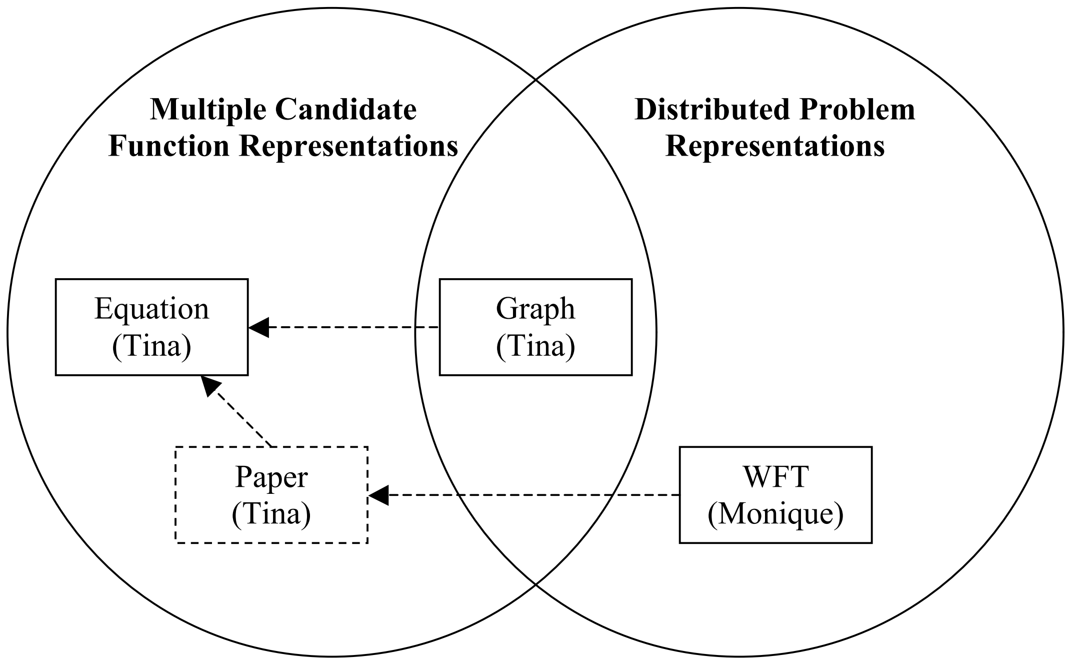

Figure 11. Configuration of Representations and Participants in Episode Four

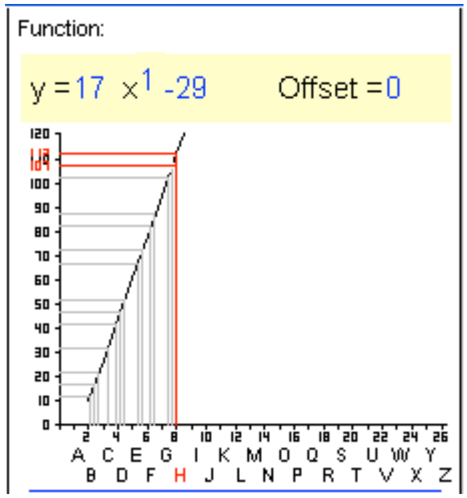

Figure 12. Inverse function bug in the Graph

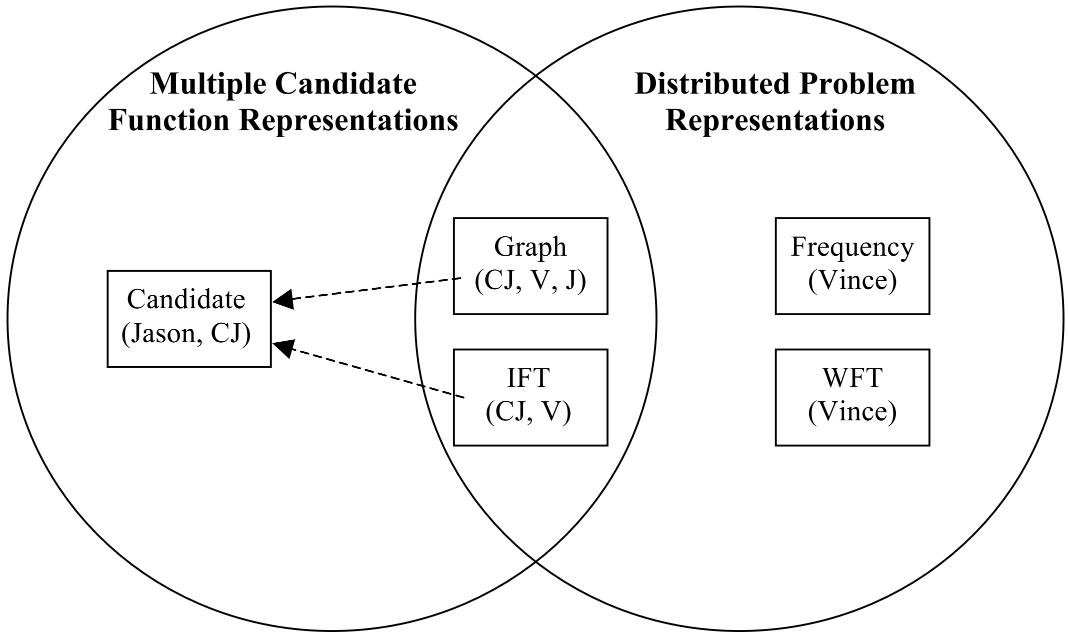

Figure 13. Configuration of Representations and Participants in Episode Five

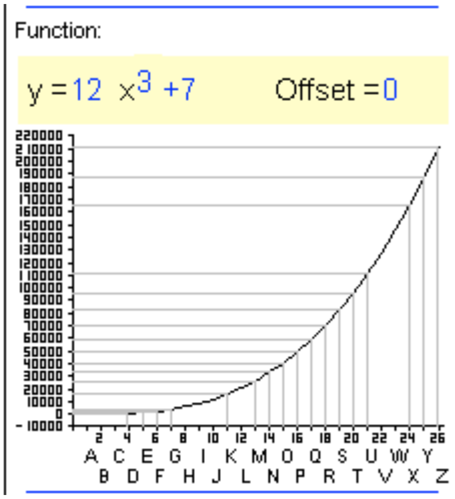

Figure 14a. Jason's view of the Equation and Graph

1	19	14	32,935
2	103	15	40,507
3	331	16	49,159
4	775	17	58,963
5	1,507	18	69,991
6	2,599	19	82,315
7	4,123	20	96,007
8	6,151	21	111,139
9	8,755	22	127,783
10	12,007	23	146,011
11	15,979	24	165,895
12	20,743	25	187,507
13	26,371	26	210,919

Frequency Table:

X	Y	Count	X	Y	Count
	15,957	1		187,485	1
	32,913	1		210,897	2
	753	1		49,137	2
	165,873	1		82,293	2
	26,349	1		111,117	3
	95,985	1		-3	3
	4,101	1		2,577	5
	69,969	1		58,941	6

Figure 14b. Vince's view of the Frequency and Function Tables

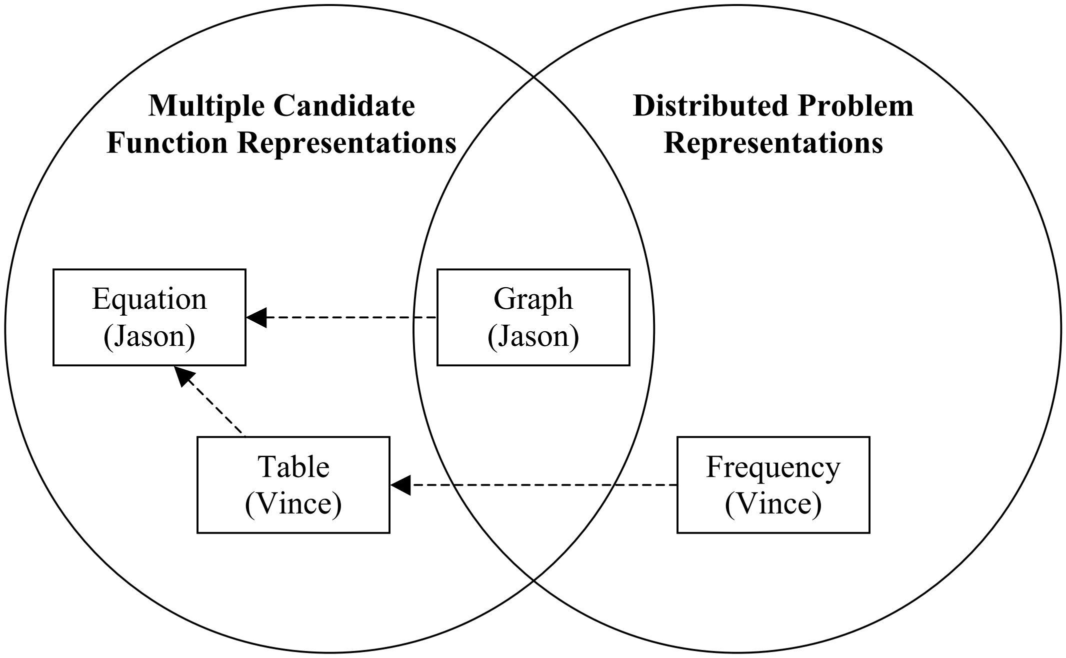

Figure 15. Configuration of Representations and Participants in Episode Six

Function Table:

X	Y	X	Y
1	24	14	52,141
2	157	15	64,130
3	518	16	77,829
4	1,221	17	93,352
5	2,380	18	110,813
6	4,109	19	130,326
7	6,522	20	152,005
8	9,733	21	175,964
9	13,856	22	202,317
10	19,005	23	231,178
11	25,294	24	262,661
12	32,837	25	296,880
13	41,748	26	333,949

Frequency Table:

X	Y	Count	X	Y	Count
	296,876	1		52,137	2
	153	1		175,960	2
	77,825	1		110,809	4
	4,105	1		64,126	4
	1,217	1		130,322	4
	231,174	1		20	7
	13,852	1		2,376	7
	6,518	2		9,729	8
	32,833	2		152,001	8
	41,744	2			

Figures 16a and b. “Close” values in the Function and Frequency Tables.

Table 1. Decryption tasks[13] and solution strategies for Group One.

Task	Encoding Function	Drew Function Inferences from Candidate Representations Only	Drew Function Inferences from Links between Candidate and Ciphertext Representations
1	$4x - 6$		
2	$-2x$		
3	$4x + 7$	Yes (Solved)	
4	$-4x + 7$	Yes (Solved)	
5	$6x - 14$	Yes (Solved)	
6	$-6x + 14$	Yes	
7	$6x - 14$		Yes (Solved)
8	$x^2 + 7$		Yes (Solved)
9	$2x^2 + 7$		Yes (Solved)
10	$2x$, offset=3		Yes (Solved)
11	$-2x + 1$, offset=2		Yes (Solved)
12	$-x^2-1$, offset=-2		Yes (Solved)
13	$37x^3 – 51$, offset=-14	Yes	
15	$-20x^3 + 16$, offset=-1		Yes (Solved)

[13] Gaps in these sequences of tasks indicate instances (task 14 for Group One, and tasks 1, 3, and 13 for Group Two) in which the group downloaded and worked on a code at least briefly, but the data available were inadequate for analysis of the strategies employed.

Table 2. Decryption tasks and solution strategies for Group Two.

Task	Encoding Function	Drew Function Inferences from Candidate Representations Only	Drew Function Inferences from Links between Candidate and Ciphertext Representations
2	$-2x^2 - 2$		
4	$5x + 7$	Yes	
5	$2x$	Yes (Solved)	
6	$-2x$	Yes (Solved)	
7	$4x + 7$	Yes (Solved)	
8	$-4x + 7$	Yes (Solved)	
9	$5x$, offset: 5	Yes	
10	$-2x + 1$, offset: 2	Yes	
11	$5x+21$	Yes	
12	$5x^3 + 6$, offset: 4	Yes	
14	$-ax^2 + b$	Yes	
15	$-3x^2 – 1$, offset		Yes
16	$12x^3 – 15$, offset: 13		Yes (Solved)
17	$19x^3 + 1$, offset: 5		Yes (Solved)